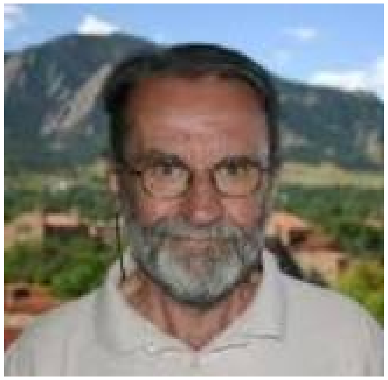
Lecture 2 (part I):

Non-Definability in First Order Logic

and

Monadic Second Order Logic

Ehrenfeucht-Fraïssé Games and Hintikka formulas

		
R. Fraïssé	A. Ehrenfeucht	J. Hintikka
1920-2008	1932-	1929-2015

Their work is from the 1950ties

Tools to Show Non-Definability

- Compactness of First Order Logic
- Ehrenfeucht-Fraïssé Games
- Translation Schemes and transductions
- Feferman-Vaught Theorem for sums
- $0 - 1$ Laws

Proving non-definability

The class of τ-structures of finite even cardinality, $EVEN(\tau)$, is *not definable* in First Order Logic, (not even in Monadic Second Order Logic):

- For FOL: use compactness. Every formula true in all finite even structures has an infinite model.
- For FOL (restricted to finite structures): use Pebble Games (Ehrenfeucht-Fraïssé Games)
- For $MSOL$: use Pebble Games adapted to $MSOL$.

Similarly, $DisPath(n)$ is not FOL-definable even for $n = 1$.

Compactness of FOL

Recall:
Σ is satisfiable if there is a τ-structure $\mathcal{A}$ such that $\mathcal{A} \models \Sigma$.

Theorem:[Gödel-Mal'cev]

Let Σ be an infinite set of $FOL(\tau)$-sentences.

Σ is satisfiable iff every finite subset $\Sigma_0 \subseteq \Sigma$ is satisfiable.

This theorem was stated and proved in Logic for CS for Propositional Logic.

This theorem was stated, but probably not proved in Logic for CS for First Order Logic.

The proof for FOL is very similar to the one for Propositional Logic.

Using Compactness

Let ϕ_n be the sentence which says that the universe contains at least n elements.

Let Σ_{even} consist of

$$\{(\phi_{2n+1} \rightarrow \phi_{2n+2}) : n \in \mathbb{N}\}$$

All **finite** models of Σ_{even} are of even cardinality.

Assume there is ψ_{even} such that

$$\mathcal{A} \models \psi_{even} \text{ iff } \mid A \mid = 2n$$

Define

$$\Sigma_1 = \{\psi_{even}\} \cup \{\phi_n : n \in \mathbb{N}\} \cup$$

Every finite subset $\Sigma_0 \subseteq \Sigma_1$ is satisfiable (by a finite model of even cardinality).

But Σ_1 has no model, **contradicting compactness**.

$MSOL$ is not compact

Let $\tau_{a,b} = \tau_{graph} \cup \{a, b\}$ be the vocabulary of graphs with two constants.

In $MSOL(\tau_{a,b})$ we have a formula ϕ_{conn} which says that the graph is connected.

Let $\psi_n(a, b)$ say that the shortest path between a, b is of length n.

This is in $FOL(\tau_{a,b})$.

Now every finite subset of

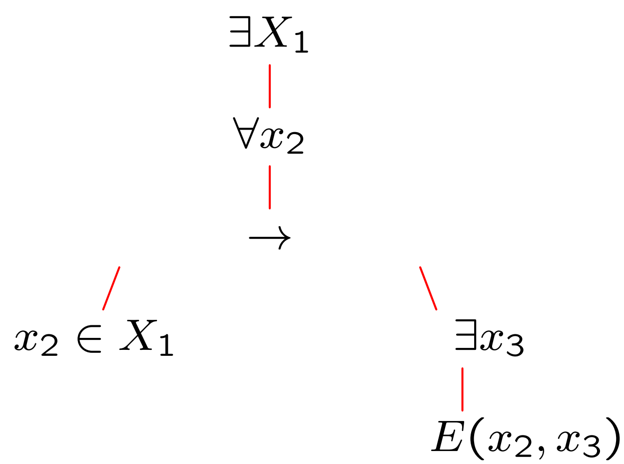
$$\Sigma = \{\phi_{conn} \cup \{\psi_n(a, b) : n \in \mathbb{N}\}$$

is satisfiable, but Σ is not.

Quantifier rank of a formula, I

We write a formula ϕ as a tree:

$$\exists X_1 \forall x_2 \, (x_2 \in X_1 \to \exists x_3 E(x_2, x_3))$$



The quantifier rank is biggest number of quantifiers one can find along a path in this tree.

Here it is 3.

Quantifier rank of a formula, II

- For formulas in prenex normal form
  the quantifier rank equals the number of quantifiers.

- If we reuse variables, the quantifier rank can be smaller than the number of quantifiers used in prenex normal form.

$$\forall x_1 \, (\exists x_2 E(x_1, x_2) \land \exists x_2 \neg E(x_1, x_2))$$

Quantifier rank 2

$$\forall x_1 \exists x_2 \exists x_3 \, (E(x_1, x_2) \land \neg E(x_1, x_3))$$

Quantifier rank 3

Ehrenfeucht-Fraïssé Games, I

Given two τ-structures $\mathcal{A}_0$ and $\mathcal{A}_1$ and their powersets $P(A_0)$ and $P(A_1)$.

Two players I (spoiler), II (duplicator)

k numbered pebbles for each structure

Two kind of moves: Set- and point-moves

Play for n moves

i-th move:
I chooses $\alpha \in \{0, 1\}$ and put pebble on an
element in $P(A_\alpha)$ (Set-move) or
in A_α (point move).
II puts corresponding pebble on set or point.

Ehrenfeucht-Fraïssé Games, II

After n moves we have from $\mathcal{A}_0$

$$A^0_1, a^0_2, A^0_3, \ldots, a^0_{n-1}, A^0_n$$

and from $\mathcal{A}_1$

$$A^1_1, a^1_2, A^1_3, \ldots, a^1_{n-1}, A^1_n$$

These two sequences are *locally isomorphic* if for all j, k

$$a^0_k \in A^0_j \text{ iff } a^1_k \in A^1_j$$

and for each m-ary $R \in \tau$ and $j_1, j_2, \ldots j_m$

$$R^{\mathcal{A}_0}(a^0_{j_1}, a^0_{j_2}, \ldots, a^0_{j_m}) \text{ iff } R^{\mathcal{A}_1}(a^1_{j_1}, a^1_{j_2}, \ldots, a^1_{j_m})$$

Lemma: Two sequences in $\mathcal{A}_0$ and $\mathcal{A}_1$ respectively

$$A^0_1, a^0_2, A^0_3, \ldots, a^0_{n-1}, A^0_n$$

and

$$A^1_1, a^1_2, A^1_3, \ldots, a^1_{n-1}, A^1_n$$

are locally isomorphic iff for all quantifierfree formulas B we have

$$\mathcal{A}_0 \models B(A^0_1, a^0_2, A^0_3, \ldots, a^0_{n-1}, A^0_n)$$

iff

$$\mathcal{A}_1 \models B(A^1_1, a^1_2, A^1_3, \ldots, a^1_{n-1}, A^1_n)$$

Proof:
Use induction over the construction of B.

Ehrenfeucht-Fraïssé Games, III

Winning the game:

II wins if the correspondence on the pebbles induces a local isomorphism (including the sets).

Theorem: (Ehrenfeucht-Fraïssé, 1953/61)
II has a winning strategy for the k-pebble n-moves game on $\mathcal{A}_0$ and $\mathcal{A}_1$ iff they satisfy the same $MSOL(\tau)$-sentences with k variables and quantifier depth n.

If no set-moves are played this holds for $FOL(\tau)$.

We write $\mathcal{A}_0 \sim_{k,n}^{MSOL} \mathcal{A}_1$ iff
II has a winning strategy in the game with set moves and
$\mathcal{A}_0 \sim_{k,n}^{FOL} \mathcal{A}_1$ in the game without set moves.

Winning strategies, I

A winning **strategy** is a function which takes a position of length n

$$A^0_1, a^0_2, A^0_3, \ldots, a^0_{n-1}, A^0_n$$

$$A^1_1, a^1_2, A^1_3, \ldots, a^1_{n-1}, A^1_n$$

from $\mathcal{A}_0$ and $\mathcal{A}_1$ respectively
together with a move of player I, say
$X^i_{n+1} \in \{a^i_{n+1}, A^i_{n+1}\}$ as input and returns
$X^{1-i}_{n+1} \in \{a^{1-i}_{n+1}, A^{1-i}_{n+1}\}$ as output such that

$$A^0_1, a^0_2, A^0_3, \ldots, a^0_{n-1}, A^0_n, X^0_{n+1}$$

$$A^1_1, a^1_2, A^1_3, \ldots, a^1_{n-1}, A^1_n, X^1_{n+1}$$

is a winning position
(if it exists, else it is undefined).

Winning strategies, II

Proposition:

$\mathcal{A}_0 \sim_{k,n}^{FOL} \mathcal{A}_1$ and $\mathcal{A}_0 \sim_{k,n}^{MSOL} \mathcal{A}_1$

are *equivalence relations* between τ-structures.
I.e., they are **symmetric, reflexive** and **transitive**.

Proof:
Reflexivity: Copy literally
Symmetry: The structures play exchangeable roles (but not the players)
Transitivity: Play on the intermediate board

Winning EF-Games, I

$\tau = \emptyset$

$$\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet$$

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$\tau = \{R_2\}$, linear orders

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Winning EF-Games, II

Theorem:

Let $\tau = \emptyset$.
For two sets $\mathcal{A}_0$ and $\mathcal{A}_1$
of size m_0 and m_1 respectively,
we have $\mathcal{A}_0 \sim^{FOL}_{k,n} \mathcal{A}_1$
(in the game without set moves)

iff

$m_0 = m_1$ or
$k \leq m_0$ and $k \leq m_1$

Winning EF-Games, III

Theorem:

Let $\tau = \{R_2\}$.
For two cycle graphs $\mathcal{G}_0$ and $\mathcal{G}_1$
of size v_0 and v_1 respectively,
we have $\mathcal{G}_0 \sim_{k,n}^{FOL} \mathcal{G}_1$
(in the game without set moves)

provided

$v_0 = v_1$ or
$2^k \leq v_0$ and $2^k \leq v_1$

Does the converse hold ?

Ehrenfeucht-Fraïssé Games, IV

Theorem: (Feferman, Vaught, 1956)

If $\mathcal{A}_0 \sim_{k,n}^{MSOL} \mathcal{B}_0$ and $\mathcal{A}_1 \sim_{k,n}^{MSOL} \mathcal{B}_1$
then $\mathcal{A}_0 \sqcup \mathcal{A}_1 \sim_{k,n}^{MSOL} \mathcal{B}_0 \sqcup \mathcal{B}_1$

Theorem: (Feferman, Vaught, 1956)

If $\mathcal{A}_0 \sim_{k,n}^{FOL} \mathcal{B}_0$ and $\mathcal{A}_1 \sim_{k,n}^{FOL} \mathcal{B}_1$
then $\mathcal{A}_0 \times \mathcal{A}_1 \sim_{k,n}^{FOL} \mathcal{B}_0 \times \mathcal{B}_1$

The same holds for "gluing" operations.

Winning EF-Games, IV

Theorem:

Let $\tau = \{R_2\}$.
Let $\mathcal{G}_0$ consist of one cycle of size 2^k
and $\mathcal{G}_1$ consist of two cycles of size 2^k.

Then we have $\mathcal{G}_0 \sim_{k,n}^{FOL} \mathcal{G}_1$
(in the game without set moves)

Corollary:
Connectivity is not FOL-definable in the language of graphs.

Winning EF-Games, V

First we play the game for $MSOL$ for $\tau = \emptyset$.

A_0 is a set of 2^n elements A_1 is a set of $2^n - 1$ elements

How many moves does player I need to win?

C_n is the undirected graph with n vertices which is connected and 2-regular.

$\mathcal{A}_0$ is the graph C_{2^n} $\mathcal{A}_1$ is the graph $C_{2^{n-1}}$ elements

How many moves does player I need to win?

Winning EF-Games, VI

The rôle of the pebbles.

How long can we play (without set moves) with **two** pebbles?

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$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

How long can we play with **three** pebbles?

Lecture 2 (part II)

Non-Definability in First Order Logic

and

Monadic Second Order Logic

Ehrenfeucht-Fraïssé Theorem

Hintikka Formulas

Ehrenfeucht-Fraïssé Theorem, I

Theorem:(Easy part)
Assume there is a $MSOL(\tau)$-sentence ϕ with k variables and quantifier depth n in Prenex Normal Form such that $\mathcal{A}_0 \models \phi$ and $\mathcal{A}_1 \models \neg\phi$.

Then I has a winning strategy for the k-pebble n-moves game on $\mathcal{A}_0$ and $\mathcal{A}_1$.

Ehrenfeucht-Fraïssé Theorem, II

We first assume that there infinitely many pebbles.

We write ϕ and $\neg\phi$ in Prenex Normal Form:

$$\phi = \exists X_1 \exists x_2 \forall X_3 \exists x_4 \ldots \exists x_{n-1} \exists X_n$$
$$B(X_1, x_2, \ldots, x_{n-1}, X_n)$$
$$\neg\phi = \forall X_1 \forall x_2 \exists X_3 \forall x_4 \ldots \forall x_{n-1} \forall X_n$$
$$\neg B(X_1, x_2, \ldots, x_{n-1}, X_n)$$

where B is without quantifiers.

We can read from the quantifier prefix
a winning strategy.

Ehrenfeucht-Fraïssé Theorem, III

Assume $\mathcal{A}_0 \models \phi$ and $\mathcal{A}_1 \models \neg\phi$.
Player I follows the existential quantifiers.

Player I picks in $\mathcal{A}_0$ a set A_1 such that

$$\mathcal{A}_0, A_1^0 \models \exists x_2 \forall X_3 \exists x_4 \ldots \exists x_{n-1} \exists X_n$$
$$B(A_1^0, x_2, \ldots, x_{n-1}, X_n)$$

Ehrenfeucht-Fraïssé Theorem, IV

Whatever player II picks as A_1^1

$$\mathcal{A}_1, A_1^1 \models \forall x_2 \exists X_3 \forall x_4 \ldots \forall x_{n-1} \forall X_n$$
$$\neg B(A_1^1, x_2, \ldots, x_{n-1}, X_n)$$

Next player I picks an element a_2^0 in $\mathcal{A}_0$ such that

$$\mathcal{A}_0, A_1^0, a_2^0 \models \forall X_3 \exists x_4 \ldots \exists x_{n-1} \exists X_n$$
$$B(A_1^0, a_2^0, \ldots, x_{n-1}, X_n)$$

Whatever player II picks as a_2^1

$$\mathcal{A}_1, A_1^1, a_2^1 \models \exists X_3 \forall x_4 \ldots \forall x_{n-1} \forall X_n$$
$$\neg B(A_1^1, a_2^1, \ldots, x_{n-1}, X_n)$$

Now player I picks in $\mathcal{A}_1$ a set A_3^1 such that

$$\mathcal{A}_1, A_1^1, a_2^1, A_3^1 \models \forall x_4 \ldots \forall x_{n-1} \forall X_n$$
$$\neg B(A_1^1, a_2^1, A_3^1, \ldots, x_{n-1}, X_n)$$

and so on..........

Ehrenfeucht-Fraïssé Theorem, V

Finally the outcome is from $\mathcal{A}_0$

$$A_1^0, a_2^0, A_3^0, \ldots, a_{n-1}^0, A_n^0$$

and from $\mathcal{A}_1$

$$A_1^1, a_2^1, A_3^1, \ldots, a_{n-1}^1, A_n^1$$

such that

$$\mathcal{A}_0 \models B(A_1^0, a_2^0, A_3^0, \ldots, a_{n-1}^0, A_n^0)$$

and

$$\mathcal{A}_1 \models \neg B(A_1^1, a_2^1, A_3^1, \ldots, a_{n-1}^1, A_n^1)$$

which shows that player $\mathbf{I}$ wins, as this cannot be a local isomorphism

(We need a Lemma on local isomorphisms and quantifierfree formulas)

How many non-equivalent formulas? FOL atomic case

Assume we have (first order) variables

$$x_1, x_2, \ldots, x_v$$

This gives $\binom{v}{2} + \binom{v}{1} = O(v^2)$ many instances of $x_i = x_j$ with $i \leq j$.

For a r-ary relation symbol R we get r^v many instances of $R(x_{j_1}, x_{j_2}, \ldots, x_{j_r})$.

If we allow $c_1, c_2, \ldots, c_{v'}$ constants the numbers become $O((v+v')^2)$ and $r^{v+v'}$ respectively.

Proposition:
For a fixed finite relational vocabulary τ with constants and v first order variables, there are a finite number of atomic formulas $\alpha^{FOL}_{\tau,v}$.

How many non-equivalent formulas?
$MSOL$ atomic case

Assume we have first and second order variables

$$x_1, x_2, \ldots, x_{v_1}, U_1, U_2, \ldots, U_{v_2}$$

This gives
$O(v_1^2)$ many instances of $x_i = x_j$ with $i \leq j$
and $v_1 \cdot v_2$ many instances of $x_i \in U_j$.

For a r-ary relation symbol R we get r^v many instances of $R(x_{j_1}, x_{j_2}, \ldots, x_{j_r})$.
If we allow $c_1, c_2, \ldots, c_{v_3}$ constants the numbers become $\binom{v_1+v_3}{2}$, $(v_1 + v_3)v_2$ and $r^{v_1+v_3}$ respectively.

Proposition:
For a fixed finite relational vocabulary τ with constants and v first order variables, there are a finite number of atomic formulas $\alpha^{MSOL}_{\tau,v}$.

How many non-equivalent formulas? Quantifierfree case

For quantifierfree formulas we only count formulas in CNF.

There are $2^{\alpha^{FOL}_{\tau,v}}$, resp. $2^{\alpha^{MSOL}_{\tau,v}}$ many disjunctions

$$\bigvee_{j=1}^{2^{\alpha^{FOL}_{\tau,v}}} (\neg)^{\nu(j)} A_j$$

where A_j ranges over atomic formulas.

Hence we have (at most) $2^{2^{\alpha^{FOL}_{\tau,v}}}$ many formulas in CNF.

Proposition:
For a fixed finite relational vocabulary τ with constants and v first order variables, there are a finite number of atomic formulas $\beta^{FOL}_{\tau,v}$ and $\beta^{MSOL}_{\tau,v}$, respectively.

How many non-equivalent formulas?
Quantifiers I: PNF

Counting quantified formulas is a bit more tricky.
We can assume that the formulas are in

Prenex Normal Form

But then variables are NOT reused.

So for each CNF formula with v variables there are $3^v \cdot v!$ many quantifier prefixes
($\exists, \forall$, not quantified).

This gives at most

$$3^v \cdot v! \cdot \beta\tau, v^{FOL}$$

many prenex normal form formulas.

How many formulas are there ?
Quantifiers II: quantifier rank

Theorem:
For each τ and $v = v_1 + v_2$ many variables

$$x_1, x_2, \ldots, x_{v_1}, U_1, U_2, \ldots, U_{v_2}$$

there are only $\gamma^{MSOL}_{\tau,v,q}$ many formulas of quantifier rank q.

Proof: We estimate this number by induction over q for $MSOL$.

For $q = 0$ we have at most γ many formulas with $\gamma_0 = \beta\tau, v^{MSOL}$.

Treating them as atomic formulas we have $2v$ many ways of adding one quantifier, and hence at most

$$\gamma^{MSOL}_{\tau,v,q+1} = \gamma_{q+1} = 2^{2^{2v \cdot \eta q}}$$

many formulas of rank $q + 1$.

How many formulas are there ?
Quantifiers II: quantifier rank

How many non-equivalent formulas
are there really?

Exact estimates to the best of our knowledge unknown.

Hintikka formulas, I

τ is a finite, relational vocabulary.

We denote by $Fm^{MSOL}_{k,q}(\tau)$ the set of $MSOL(\tau)$ formulas such that the variables are among

$$x_1, \ldots, x_k, U_1, \ldots, U_k$$

and each formula has quantifier rank atmost q.

Similarly with $Fm^{FOL}_{k,q}(\tau)$.

Definition:
ϕ and ψ are (finitely) equivalent if the have the same (finite) models.
Free variables are uninterpreted constants

Note: There are, up to logical equivalence infinitely many formulas in three variables (use repeated quantification).

The boolean algebra $Fm_{k,q}(\tau)$, I

Proposition:
There are, up to (finite) equivalence, only finitely many formulas in $Fm_{k,q}(\tau)$.

If ϕ and ψ have only infinite models, they are finitely equivalent (**false**).
There are fewer formulas for finite equivalence.
The number of equivalence classes is growing very fast.

Proposition:
$Fm_{k,q}(\tau)$ is closed under conjunction $\wedge$,
disjunction $\vee$ and negation $\neg$,
i.e. it forms a finite *boolean algebra*.

The boolean algebra $Fm_{k,q}(\tau)$, II

The formula $\exists x(x \neq x)$ is the *minimal element*.

The formula $\exists x(x = x)$ is the *maximal element*.

A formula ϕ is an *atom*, if

- it is not (finitely) equivalent to $\exists x(x \neq x)$,
- but for each ψ either $\phi \wedge \psi$ is equivalent to ϕ or to $\exists x(x \neq x)$.

Hintikka formulas, II

We denote by $\mathcal{B}_{k,q}(\tau)$ and $\mathcal{B}^f{}_{k,q}(\tau)$ the
finite boolean algebra of $Fm_{k,q}^{MSOL}(\tau)$
up to equivalence and finite equivalence, resp.
The elements are denoted by $\bar{\phi}$.

The set of atoms in $\mathcal{B}_{k,q}(\tau)$ and $\mathcal{B}^f{}_{k,q}(\tau)$ is denoted by $\mathcal{H}_{k,q}(\tau)$ and $\mathcal{H}^f{}_{k,q}(\tau)$.

The formulas ϕ with $\bar{\phi} \in \mathcal{H}_{k,q}(\tau)$ ($\bar{\phi} \in \mathcal{H}^f{}_{k,q}(\tau)$) are called *Hintikka formulas*.

Hintikkika formulas, III

Proposition:

(i) Every sentence $\phi \in Fm_{k,q}(\tau)$ is equivalent to the disjunction of a unique set of
(k,q)- Hintikka sentences $\bigvee_i h_{i(\phi)}$,
with $\bar{h}_{i(\phi)} \in \mathcal{H}_{k,q}(\tau)$.

Not computable from k, q, τ and ϕ alone.

(ii) For every k, q, τ and τ-structure $\mathcal{A}$ there is a unique Hintikka sentence $h_{k,q}(\mathcal{A}) \in Fm_{k,q}(\tau)$ such that
$\mathcal{A} \models h_{k,q}(\mathcal{A})$.

(iii) Furthermore, if $\mathcal{A}$ is finite,
$h_{k,q}(\mathcal{A})$ is computable from k, q, τ and $\mathcal{A}$.

But only highly ineffective algorithms are known.

Hintikka formulas, IV

Theorem:(Ehrenfeucht-Fraïssé)

For two τ-structures $\mathcal{A}_1$ and $\mathcal{A}_2$ the following are equivalent:

(i) II has a winning strategy in the game with n moves and k point pebbles and
k set pebbles.

(ii) $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy the same sentences of $Fm_{k,m}(\tau)$.

(iii) $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy the same unique (up to equivalence) (k, m)-Hintikka sentence.

We have shown already (1) $\Rightarrow$ (3).
(2) $\Rightarrow$ (3) is trivial.
(3) $\Rightarrow$ (2) follows from the
properties of Hintikka formulas.
We are left with (3) $\Rightarrow$ (1).

Constructing the Hintikka sentence, I

Assume we have more pebbles than moves.

Let $\mathcal{A}$ be a finite τ-structure and $a_1, a_2, \ldots, a_s$ elements $\mathcal{A}$.

We define a formula $\phi(v_1, \ldots, v_s)^m_{\bar{a}}$

such that

$$\mathcal{A}, \bar{a} \models \phi(v_1, \ldots, v_s)^m_{\bar{a}}$$

and whenever

$$\mathcal{B}, \bar{b} \models \phi(v_1, \ldots, v_s)^m_{\bar{a}}$$

then player **II** has a winning strategy in the game for FOL for m more moves starting with $\mathcal{A}, \bar{a}$ and $\mathcal{B}, \bar{b}$.

$\phi(v_1, \ldots, v_k)^q_{\bar{a}}$ (i.e. $k = s, q = m$) will be a
Hintikka formula for $Fm^{FOL}_{k,q}(\tau)$.

Constructing the Hintikka sentence, II

$$\phi(v_1, \ldots, v_k)^0_{\bar{a}} :=$$

$$\left(\bigwedge\{R(v_{j_1}, \ldots, v_{j_s}) : R \in \tau, \mathcal{A}, \bar{a} \models R(v_{j_1}, \ldots, v_{j_s})\}\right)$$
$$\wedge$$
$$\left(\bigwedge\{\neg R(v_{j_1}, \ldots, v_{j_s}) : R \in \tau, \mathcal{A}, \bar{a} \models \neg R(v_{j_1}, \ldots, v_{j_s})\}\right)$$
$$\wedge$$
$$\left(\bigwedge\{v_{j_1} = v_{j_2} : j_1, j_2 \leq s \text{ and } \mathcal{A}, \bar{a} \models v_{j_1} = v_{j_2}\}\right)$$
$$\wedge$$
$$\left(\bigwedge\{v_{j_1} \neq v_{j_2} : j_1, j_2 \leq s \text{ and } \mathcal{A}, \bar{a} \models v_{j_1} \neq v_{j_2}\}\right)$$

The formula is finite, provided τ is.

Homework (compulsory)

We look at the example of a linear order with $s = 3$ and $m = 2$.

Assume $a_2 < a_1 = a_3$ in $\mathcal{A}$.

Compute the formula!

Constructing the Hintikka sentence, III

$$\phi(v_1, \ldots, v_k)^m_{\bar{a}} :=$$
$$\left(\bigwedge_{a \in A} \exists v_{s+1} \phi(\bar{v}, v_{s+1})^{m-1}_{\bar{a} \cdot a} \right) \wedge$$
$$\left(\forall v_{s+1} \bigvee_{a \in A} \phi(\bar{v}, v_{s+1})^{m-1}_{\bar{a} \cdot a} \right)$$

This is finite by the previous theorem.

We look at the example of a linear order with $s = 3$ and $m = 2$.

Assume $a_2 < a_1 = a_3$ in $\mathcal{A}$.

Compute the formula.

Constructing the Hintikka sentence, IV

We have to verify:

- $\mathcal{A}, \bar{a} \models \phi(v_1, \ldots, v_s)^m_{\bar{a}}$
- whenever $\mathcal{B}, \bar{b} \models \phi(v_1, \ldots, v_s)^m_{\bar{a}}$
  then player II has a winning strategy
  in the game for FOL for m more moves
  starting with $\mathcal{A}, \bar{a}$ and $\mathcal{B}, \bar{b}$.

Constructing the Hintikka sentence, V

- We can do ”*the same*” for $MSOL$ and even for SOL^n or SOL.
- How do we have to modify the construction of there are fewer pebbles than moves?
- What happens if play infintely long?

We shall return to these questions later.

Dense linear orders, I

We look at linear orders such that between any two distinct elements there is a third element. These are called *dense linear orders*.

Exercise:
Express this in FOL.
Show that such an order is always infinite.

There are variations:

- with/without first element.
- with/without last element.

Dense linear orders, II

Examples are

- The real numbers $\mathbb{R}$, which are uncountably infinite.
- The irrational numbers $\mathbb{I} \subseteq \mathbb{R}$, which are also uncountably infinite.
- The rationals $\mathbb{Q}$, which are countably infinite.
- The open intervals $(a, b) \subseteq \mathbb{R}$.
- The open intervals $(a, b) \subseteq \mathbb{Q}$.
- The corresponding closed intervals $[a, b]$ and the intervals $(a, b]$ and $[a, b)$.

Dense linear orders, III

There is a sentence $\phi_{cut} MSOL(\tau_{ord})$ which is true in $\mathbb{Q}$ but not in $\mathbb{R}$.

ϕ_{cut} says:

"The universe is the disjoint union of two open intervals"

Exercise:
Write down this formula.

In $\mathbb{Q}$ we take $(-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)$.

In $\mathbb{R}$ every Cauchy sequence converges, hence such a decomposition is not possible.

Dense linear orders, IV

Theorem:(Cantor ca. 1870)
Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be two dense linear orders with the same configuration of first and last elements.

Then player II has one (extendible) winning strategy WS in the FOL game for games of arbitrary finite length.

Note that this is **stronger** than the statement:
For every game length n player II has a winningstrategy WS_n.

Corollary:
No $FOL(\tau_{ord})$ sentence ϕ can distinguish
$\mathbb{Q}$ from $\mathbb{R}$, or
$(a,b] \cap \mathbb{Q}$ from $(a,b] \cap \mathbb{R}$ for $a,b \in \mathbb{Q}$,
etc...

Dense linear orders, V

Proof: (No first, no last element)

Assume we have played

$$a^0_{k_1} \leq a^0_{k_2} \leq \ldots \leq a^0_{k_m} \text{ and } a^1_{k_1} \leq a^1_{k_2} \leq \ldots \leq a^1_{k_m}$$

and player I chooses, w.l.o.g., $a^0_{m+1} = b$.

There are three cases

- $b < a^0_{k_1}$ or $a^0_{k_m} < b$.
- $b = a^0_{k_j}$ for some $j \leq m$.
- $a^0_{k_{j-1}} < b < a^0_{k_j}$ for some $j \leq m$.

In each case II can reply correspondingly.
In the last case we use density.
In the first case we use the absence of first/last elements.

Exercise:
Complete the proof also for the cases with first/last elements.