Part 5

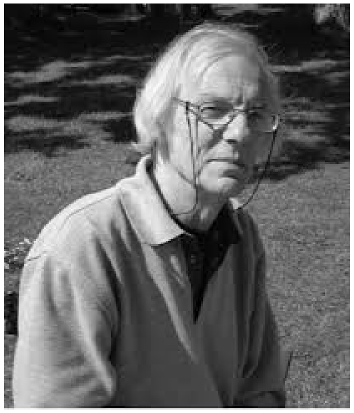
5.1:The Lindström Theorems

5.2: Craig Interpolation

Outline of Part 5.1

- Basic Properties of First order Logic
- Mostowski Quantifiers
- Infinitary Logic
- Model Theoretic Logics and How to Compare Them?
- The Lindström Theorems
- A. Robinson's Consistency Theorem

What is so special about First Order Logic FOL?



Per Lindström (1936-2009)

Basic Properties of First Order Logic FOL, I

So far we note the following BASIC properties (B0)-(B7) and (ISO) of FOL

B0: Let τ be a countable vocabulary. τ-structures are natural set interpretations of τ.

B1: Formulas have at most finitely many free individual variables and no free relation or function variables.

B2: The set of formulas form a Boolean algebra with corresponding semantics.

B3: The set of formulas are closed under existential quantification over elements of the universe of τ-structures.

Basic Properties of First Order Logic FOL, II

B4: The set of formulas is closed under substitution of relation symbols by formulas with the same number of free variables, with the corresponding interpretation of formulas.

B5: The set of formulas is closed under renaming of τ, with the corresponding invariance of the interpretation of formulas.

B6: The set of formulas is closed under relativization of formulas with the corresponding interpretation for relativized formulas.

B7: The interpretation of formulas is invariant under isomorphisms of τ-structures.

ISO: Isomorphic structures satisfy the same sentences.

Model theoretic properties of First Order Logic FOL

We have seen the following model theoretic properties.

TRANS: FOL satisfies the Fundamental Theorem of Translations and Transductions.

COMP: FOL satisfies the Compactness Theorem.
A countable set of formulas Σ is satisfiable iff every finite subset of Σ is satisfiable.

LS: FOL satisfies the Löwenheim-Skolem Theorem.
A countable set of formulas Σ is satisfiable iff it has a countable or finite model.

AXIOM: The set of formulas is recursive (computable) and the set of valid formulas is recursively enumerable.

EF: Two structures satisfy the same formulas of quantifier rank k iff the Ehrenfeucht-Fraïssé Game of length k has a Winning Strategy for Player II (The Duplicator).

Mostowski quantifiers

Let $Q_\alpha x \phi(x)$ stand for "There are at least $\aleph_\alpha$ many elements x which satisfy $\phi(x)$".

Let FOL(Q_α) be the logic obtained from FOL by adding this quantifier in the inductive definition of the formation rules of FOL, and giving it its appropriate semantics.

- For $\alpha = 0$ this just says that there are infinitely many x.
  FOL(Q_0) is **is not compact**, does not satisfy COMP.

- FOL(Q_1) **is compact**, but only for countable sets of sentences. It also satisfies AXIOM, but not LS.

  This was shown by Fuhrken and Keisler in 1970

Infinite conjunctions and disjunctions

We denote by $\mathcal{L}_{\kappa,\lambda}$ the logic obtained from FOL by closing its inductive definition under

- **conjunctions and disjunction of size less than κ**, and
- **quantifier strings of size less than λ**,
- and keeping the number of free variables in formulas **finite**.
- $\mathcal{L}_{\omega,\omega} = \text{FOL}$.
- $\mathcal{L}_{\omega_1,\omega}$ satisfies LS.

This was obeserved by several people in the late 1950ties (C. Karp, E. Engeler, and others).

Lindström quantifiers

Let τ and σ be purely relational vocabularies,
and let $\mathcal{K}$ be a class of σ-structures closed under σ-isomorphisms.

Let $\Phi = \langle \phi(\bar{x}); \psi_1(\bar{y}_1), \ldots, \psi_k(\bar{y}_k) \rangle$ be a $\tau - \sigma$-translation scheme, with all the free variables $\bar{z} = (\bar{x}, \bar{y}_1, \ldots, \bar{y}_k)$ distinct.

We build an extension FOL($\mathcal{K}$) of FOL by adding the following formation rule for a Lindström quantifier:

If Φ is a $\tau - \sigma$-translation scheme with all the formulas in FOL($\mathcal{K}$)(τ) then

$$Q_{\mathcal{K}} \bar{z} \Phi$$

is a formula of FOL($\mathcal{K}$)(τ), with semantics

$$\mathcal{A} \models Q_{\mathcal{K}} \bar{z} \Phi \text{ iff } \Phi^*(\mathcal{A}) \in \mathcal{K}$$

Similarly, we can add countably many formation rules for classes of σ_i-structures $\mathcal{K}_i$, with $i \in \mathbb{N}$.

Predicate Transformers

Quantifiers correspond to boolean queries.

An k-ary Predicate Transformer corresponds to a query with k variables.

- Let $\mathcal{F}$ be a function which assocites with a σ-structure $\mathcal{A}$ a k-ary relation on $\mathcal{A}$.
- We build an extension FOL($\mathcal{F}$) of FOL by adding the following formation rule for a Lindström quantifier:
- Let Φ be a $\tau - \sigma$-translation scheme. Assume the free variables in Φ are in $\bar{x}$ and $y_1, \ldots, y_k$ are variables not in $\bar{x}$.

  Then $P_F \bar{x}, \bar{y} \Phi$ is a formula with bound variables $\bar{x}$ and free variables $\bar{y}$.
- Let $\bar{a}$ be an assignment for $\bar{y}$.

$$\mathcal{A}, \bar{a} \models P_F \bar{x}, \bar{y} \Phi \text{ iff } \bar{a} \in \mathcal{F}(\Phi^*(\mathcal{A}))$$

Similarly, we can add countably many formation rules for classes of σ_i-structures $\mathcal{K}_i$, with $i \in \mathbb{N}$.

Examples of Lindström quantifiers

- Mostowski quantifiers are Lindström quantifiers (Exercise).

- For $\sigma = \{R\}$ consisting of a single binary relation symbol, and $\mathcal{K} = \text{WO}$ the class of well-orderings, we have

$$\mathfrak{A} \models Q_{\text{WO}} x, y \phi(x, y) \text{ iff } \langle A, \phi^A \rangle \in \text{WO}$$

  i.e., $\phi(x, y)$ defines a well-ordering on A.

- Similarly for $\mathcal{K} = \text{CONN}$ be the class of connected binary relations.

- Similarly for $\mathcal{K} = \text{FCONN}$ be the class of finite connected binary relations.

- Similarly for $\mathcal{K} = \text{TREE}$ be the class of binary relations which form a tree.

Examples of predicate transformers

- Let σ consist of one binary relation symbol S, and $\mathcal{F}$ map the interpretation of S into its transitive closure.

- Let σ consist of two binary relation symbol S_1, S_2, and $\mathcal{F}$ map the interpretation of S_1 and S_2 into the symmetric transitive closure of the compositions of the interpretations of S_1 and S_2.

- More generally,
  let Π be a PROLOG program over σ which defines a relation R, and
  let $\mathcal{F}$ map the interpretations of σ into the interpetaion of R.

(Abstract) Extensions of FOL

A variable assignment into a set A is a function $z : \text{Var} \to A$.
We denote by $\text{Ass}(A)$ the set of such assignments.

A model theoretic logic $\mathcal{L}$ is given as follows:

For each purely relational countable vocabulary τ we have

- a class of τ-structures $\text{Str}(\tau)$;
- a set of formulas $\text{Form}(\tau)$, and
- a satisfaction relation $M \subseteq \text{Str}(\tau) \times \text{Form}(\tau) \times \text{Ass}(A)$.
  We write also $\mathfrak{A} \models_z \phi$ for $M(\mathfrak{A}, \phi, z)$.
  For ϕ without free variables we write $\text{Mod}(\phi)$ for the set

$$\text{Mod}(\phi) = \{\mathfrak{A} \in \text{Str}(\tau) : \mathfrak{A} \models_z \phi\}$$

We add indices of several logics are considered together.

Comparing logics

Let $\mathcal{L}_1, \mathcal{L}_2$ be two logics.

- $\mathcal{L}_2$ is at least as expressive as $\mathcal{L}_1$, $\mathcal{L}_1 < \mathcal{L}_2$, if for every $\phi \in \mathrm{Form}_1(\tau)$ there is $\psi \in \mathrm{Form}_2(\tau)$ such that $\mathrm{Mod}_1(\phi) = \mathrm{Mod}_2(\psi)$

- $\mathcal{L}_2$ is equally expressive as $\mathcal{L}_1$, $\mathcal{L}_1 \sim \mathcal{L}_2$, if both $\mathcal{L}_1 < \mathcal{L}_2$ and $\mathcal{L}_2 < \mathcal{L}_1$ hold.

By our definitions, every logic $\mathcal{L}$ is at least as expressive as FOL.

FOL augmented by quantifiers $\exists_{\leq n} x \phi(x)$,
"there exists at least n many x such that $\phi(x)$",
is equally expressive as FOL.

Lindström Logics

A Lindström Logic is a logic of the form FOL$(\mathcal{K}_i, \mathcal{F}_j), i, j \in \mathbb{N}$.

- Every countable logic with the basic closure properties can be turned into a Lindström Logic.

- Let $\mathcal{L}$ be such a logic.
  Let $\mathcal{K}_i$ be all ist definable classes of structures, and $\mathcal{K}_j$ be all its definable predicate transformers. Then FOL$(\mathcal{K}_i, \mathcal{F}_j$ has the same expressive power as $\mathcal{L}$.

- We need here that every $\mathcal{L}$-formula has only a finite number of free variables, but also that we have access to countably many individual variables.

More logics which are strictly more expressive on infinite structures than FOL besides $\mathrm{MSOL}, \mathrm{CMSOL}, \mathrm{SOL}$.

To be discussed further on the blackboard (whiteboard) ......

- FOL with a quantifier $\exists^{fin} x \phi(x)$, "there are finitely many x such that ϕ".
  This can also be expressed as $\neg Q_0 x \phi(x)$.

- FOL with a quantifier $\exists^{count} x \phi(x)$, "there are at most countably many x such that ϕ".
  This can also be expressed as $\neg Q_1 x \phi(x)$.

- FOL with a quantifier $Haertig x, y(\phi(x), \psi(x))$, "the sets defined by $\phi(x)$ and $\psi(x)$ have the same cardinality".
  Herre, Heinrich, Michal Krynicki, Alexandr Pinus, and Jouko V a an aen.
  "The H artig quantifier: a survey." The Journal of symbolic logic 56, no. 4 (1991): 1153-1183.

Even more logics which are strictly more expressive on infinite structures than FOL besides MSOL, CMSOL, SOL.

- FOL with partially ordered qunatifier prefix.
  Walkoe, Wilbur John. "Finite partially-ordered quantification." The Journal of Symbolic Logic 35.4 (1970): 535-555.
  Blass, Andreas, and Yuri Gurevich. "Henkin quantifiers and complete problems." Annals of Pure and Applied Logic 32 (1986): 1-16.

- Logic with countable conjunctions and disjunctions, $\mathcal{L}_{\omega_1,\omega}$.

End of Session 9 (December 30, 2021)

Begin of Session 10 (January 6, 2022)

The Lindström Theorems (1966, 1969)

- We first construct the **magic model**.
- We will use many techniques introduced in before in these lectures:
  - Ehrenfeucht-Fraïssé Game.
  - Vaught's Trick:
    Let Σ be a set of τ-sentences of FOL such that Σ has no finite model, and for some infinite cardinal κ all models of Σ of cardinality κ, are isomorphic. Then Σ is complete, i.e., for every τ-sentence $\phi \in$ FOL either $\Sigma \models \phi$ or $\Sigma \models \neg\phi$.
  - Cantor's Trick

Assume a logic $\mathcal{L}_2$ satisfies Löwenheim-Skolem Theorem (LS)

and is strictly more expressive than $\mathcal{L}_1 = \mathsf{FOL}$.

Then we have: There is τ and $\phi \in \mathsf{Form}_1(\tau)$ such that for no $\theta \in \mathsf{FOL}(\tau)$ we have $\mathsf{Mod}_1(\phi) = \mathsf{Mod}_2(\psi)$. In other words, there are τ-structures $\mathfrak{A}_n, \mathfrak{B}_n$ such that

- All $\mathfrak{A}_n \models \phi$ and all $\mathfrak{B}_n \models \neg\phi$, and
- for all n we have $\mathbf{II}$ has a winning strategy in the game on $\mathfrak{A}_n$ and $\mathfrak{B}_n$ with n moves and n point pebbles.

Assume further that $\mathcal{L}_2$ satisfies the Löwenheim-Skolem Theorem (LS).

Then all the $\mathfrak{A}_n$ and $\mathfrak{B}_n$ can be assumed to be **countable**.

Constructing the magic model $\mathfrak{C}$, I

The magic vocabulary $\bar{\tau}$

- $\bar{\tau}$ has all the relation symbols needed for arithmetic, the ternary relations for addition and multiplication Add and Mult, and the binary relation symbol $\leq$ for the linear order.
- $\bar{\tau}$ has two binary relation symbols $U_1(-,-)$ and $U_2(-,-)$.
- For each relation symbol $R \in \tau$ of arity $\rho(R)$ we introduce two relation symbols R_1, R_2 of arity $\rho(R) + 1$.
- For each $k \in \mathbb{N}$ we have a $(2k+2)$-ary relation symbol $E(-,- \ldots,-)$.

Constructing the magic model $\mathfrak{C}$, II

The magic structure $\mathfrak{C}$.

Next we construct a countable $\bar{\tau}$-structure $\mathfrak{C}$ as follows:

- The universe of $\mathfrak{C}$ is $\mathbb{N}$.
- Add($\mathfrak{C}$), Mult($\mathfrak{C}$), and $\leq(\mathfrak{C})$ are given by the standard interpretations on $\mathbb{N}$.
- For each $k \in \mathbb{N}$ the sets defined by $U_1(-, k)$ and $R_1(k, -, - \ldots -)$ form a structure isomorphic to the τ-structure $\mathfrak{A}_k$, and for $U_2(-, k)$ and $R_2(k, -, - \ldots -)$ form a structure isomorphic to the τ-structure $\mathfrak{B}_k$.

Constructing the magic model $\mathfrak{C}$, III

What is true in $\mathfrak{C}$.

We note that

PEANO: the Peano axioms are true in $\mathfrak{C}$ for addition, multiplication and order.
This is an infinite recursive set of formulas.

A1: The formula $\phi^{U_1(-,k)}$, ϕ relativized to $U_1(-,k)$, holds for all k.

A2: The formula $\neg\phi^{U_2(-,k)}$, $\neg\phi$ relativized to $U_2(-,k)$, holds for all k.

Constructing the magic model $\mathfrak{C}$, IV

The magic structure $\mathfrak{C}$, continued

- For two k tuples $\bar{a}$ $\bar{b}$ and $n, \alpha \in \mathbb{N}$ we have $E(\bar{a}, \bar{b}, n, \alpha)$ iff $\mathfrak{A}_n, \bar{a}, \mathfrak{B}_n, \bar{b}$ is a winning position for the game lasting another α moves in the Ehrenfeucht-Fraïssé Game.

- In particular $E(n, \alpha)$ says that $\mathfrak{A}_n$ and $\mathfrak{B}_n$ with the empty sequence is a winning position for the came lasting another α moves, i.e., they satisfy the same sentences of quantifier rank α.

Constructing the magic model $\mathfrak{C}$, V

What is true in $\mathfrak{C}$, continued

We note that

EF: $\forall n, \alpha E(n, \alpha)$.

BACK: $\forall n \bar{a}, \bar{b}, a' \exists b' E_n(\bar{a}, \bar{b}, n, \alpha) \rightarrow E_{n+1}(\bar{a}, \bar{b}, a'b', n, \alpha - 1)$
This is an infinite recursive set of formulas, for each arity one formula

FORTH: $\forall n \bar{a}, \bar{b}, b' \exists a' E_n(\bar{a}, \bar{b}, n, \alpha) \rightarrow E_{n+1}(\bar{a}, \bar{b}, a'b', n, \alpha - 1)$
This is an infinite recursive set of formulas, for each arity one formula

Let Σ consist of the formulas $EF, FORTH, BACK, PEANO, A1, A2$.

Only $A1, A2$ are not FOL-formulas.

Non-standard elements in extensions of $\mathfrak{C}$

Let $\mathfrak{D}$ be a $\bar{\tau}$-structure with universe D, which is a model of a set Σ in $\mathcal{L}_2$ which contains $PEANO$.

An element $d \in D$ is a non-standard element if in $\mathfrak{D}$ there are infinitely many elements $c_i \in D$ such that

$$\mathfrak{D} \models \{c_i < c_{i+1} : i \in \mathbb{N}\} \cup \{c_i < d : i \in \mathbb{N}\}$$

We shall denote by $\tau arith$ the relational vocabulary of arithmetic, with addition, multiplication and order.

Case 1: Σ has no model with a non-standard element.

Assume furthermore that $\mathcal{L}_2$ satisfies AXIOM.

We proceed to show that, under this assumption, the set of true FOL-sentences in $\mathcal{N}$, the standard model of arithmetic, is decidable.

- Σ is complete for arithmetic: For each $\theta \in$ FOL(τ_{arith}) we have either $\Sigma \models \theta$ or $\Sigma \models \neg\theta$.

  This follows from an argument similar to Vaught's trick.

- As $\mathcal{L}_2$ satisfies AXIOM, and Σ is a recursive enumerable set, the result follows.

Conclusion: If $\mathcal{L}_2$ satisfies AXIOM and LS then Σ has a countable model $\mathfrak{D}$ with a non-standard element d.

Case 2: Σ has a model with a non-standard element.

We proceed to show that inside $\mathfrak{D}$ we find two structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$ defined by $U_1(-, d)$ and $U_2(-, d)$ respectively, such that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are isomorphic.

- We can start playing the Ehrenfeucht-Fraïssé Game, since in $\mathfrak{D} \models E(d, d)$.
- This says that on the structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$ we can play the Ehrenfeucht-Fraïssé Game infinitely long.
- Using BACK and FORTH we can repeat Cantor's Trick, and construct an isomorphism between $\mathfrak{A}_1$ and $\mathfrak{A}_2$.
- But by A1 and A2 we have that $\mathfrak{A}_1 \models \phi$ and $\mathfrak{A}_2 \models \neg\phi$, which contradicts that $\mathcal{L}_2$ satisfies ISO.

Conclusion: If Σ has a model with a non-standard element,
then every $\phi \in \mathcal{L}_2(\bar{\tau})$ is equivalent to a first order sentence.

The Lindström Theorems

We have therefore shown:

THEOREM: Let $\mathcal{L}$ be a logic which satisfies the basic properties BASIC, ISO, and LS.

- If additionally $\mathcal{L}$ satisfies AXIOM, then $\mathcal{L}$ is equally expressive as FOL.
- If alternatively $\mathcal{L}$ satisfies COMP, then $\mathcal{L}$ is equally expressive as FOL.

End of Session 10 in Part 5.

Session 10 was completed with the
Specker-Blatter Theorem Lecture from the
Technion Combinatorics Seminar

Begin of Session 11 in Part 5

Begin of Session 11 (January 13, 2022)

A. Robinson's Joint Consistency Theorem

Abraham Robinson (born Robinsohn) 1918-1974

Two versions of the Joint Consistency Theorem

Let τ_0, τ_1, σ be three vocabularies, of any cardinality, with $\sigma = \tau_0 \cap \tau_1$.

Let $\Sigma_0 \subseteq \mathrm{FOL}(\tau_0)$, and $\Sigma_1 \subseteq \mathrm{FOL}(\tau_1)$ be each satisfiable.

Theorem:

(i) Let $\Sigma \subseteq \mathrm{FOL}(\sigma)$ a complete (satisfiable) set of sentences, and assume that for each $i = 0, 1$ the theories $\Sigma_i \cup \Sigma$ are satisfiable.

Then $\Sigma_0 \cup \Sigma_1 \cup \Sigma$ is satisfiable.

(ii) Assume that for no formula $\phi \in \mathrm{FOL}(\sigma)$ is it the case that $\Sigma_0 \models \phi$ and $\Sigma_1 \models \neg\phi$.

Then $\Sigma_0 \cup \Sigma_1 \cup \Sigma$ is satisfiable.

It is immediate that (i) implies (ii).

(ii) implies (i) uses compactness

Proof of the Joint Consistency Theorem, I

- Our proof is model-theoretic.
- It uses Ehrenfeucht-Fraïssé Games.
- It uses compactness.
- It uses the Löwenheim-Skolem Theorem (LS).
- It has the same strategy as the proof of the Lindström Theorems.

Proof of the Joint Consistency Theorem, II

We first assume that τ_0, τ_1, σ are countable.

We build a version of the magic model:

- Let $A, B \subseteq \mathbb{N}$ be two disjoint subsets of $\mathbb{N}$.

- Let $\mathfrak{A}$ and $\mathfrak{B}$ countable τ_0-, respectively τ_1-structures with universe A and B such that
$$\mathfrak{A} \models \Sigma_0 \text{ and } \mathfrak{B} \models \Sigma_1$$

- Since Σ is complete, the σ-reducts $\mathfrak{A}|_\sigma$ and $\mathfrak{B}|_\sigma$ satisfiy the same FOL(σ)-sentences.

- We construct the magic model $\mathfrak{C}$ as in the proof of the Lindström Theorems.

Proof of the Joint Consistency Theorem, III

The vocabulary $\bar{\tau}$ of $\mathfrak{C}$ is given by:

- $\bar{\tau}$ has all the relation symbols needed for arithmetic, the ternary relations for addition and multiplication Add and Mult, and the binary relation symbol $\leq$ for the linear order.
- $\bar{\tau}$ has two unary relation symbols $U_0(-)$ and $U_1(-)$.
- For each relation symbol $R \in \tau_i$ of arity $\rho(R)$ we introduce two relation symbols R_1, R_2 of arity $\rho(R) + 1$.
- For each $k \in \mathbb{N}$ we have a $(2k+1)$-ary relation symbol $E(-, - \ldots, -)$.

Proof of the Joint Consistency Theorem, IV

The $\bar{\tau}$-structure $\mathfrak{C}$ is given by:

- The universe of $\mathfrak{C}$ is $\mathbb{N}$.
- Add($\mathfrak{C}$), Mult($\mathfrak{C}$), and $\leq(\mathfrak{C})$ are given by the i standard interpretations on $\mathbb{N}$.
- The set defined by $U_0(-)$ and $R_1(k, -, - \ldots -)$ forms a structure isomorphic to the τ_0-structure $\mathfrak{A}$, and for $U_1(-)$ and $R_2(k, -, - \ldots -)$ forms a structure isomorphic to the τ_1-structure $\mathfrak{B}$.

Proof of the Joint Consistency Theorem, IV

We note that the following holds in $\mathfrak{C}$:

PEANO: the Peano axioms are true in $\mathfrak{C}$ for addition, multiplication and order.
This is an infinite recursive set of formulas.

A0: For all $\phi \in \Sigma_0$ the formula $\phi^{U_0(-}$, ϕ relativized to $U_0(-)$.

A1: For all $\phi \in \Sigma_1$ the formula $\phi^{U_1(-}$, ϕ relativized to $U_1(-)$.

Proof of the Joint Consistency Theorem, V

- For two k tuples $\bar{a}$ $\bar{b}$ and $\alpha \in \mathbb{N}$ we have $E(\bar{a}, \bar{b}, \alpha)$ iff $\mathfrak{A}, \bar{a}, \mathfrak{B}, \bar{b}$ is a winning position for the came lasting another α moves in the Ehrenfeucht-Fraïssé Game.

- In particular $E(\alpha)$ says that $\mathfrak{A}$ and $\mathfrak{B}$ with the empty sequence is a winning position for the came lasting another α moves, i.e., they satisfy the same sentences of quantifier rank α.

Proof of the Joint Consistency Theorem, VI

We note that

EF: $\forall \alpha E(\alpha)$.

BACK: $\forall \bar{a}, \bar{b}, a' \exists b' E_n(\bar{a}, \bar{b}, \alpha) \rightarrow E_{n+1}(\bar{a}, \bar{b}, a'b', \alpha - 1)$
This is an infinite recursive set of formulas, for each arity one formula

FORTH: $\forall n \bar{a}, \bar{b}, b' \exists a' E_n(\bar{a}, \bar{b}, \alpha) \rightarrow E_{n+1}(\bar{a}, \bar{b}, a'b', \alpha - 1)$
This is an infinite recursive set of formulas, for each arity one formula

Let Σ^* consist of the formulas $EF, FORTH, BACK, PEANO, A1, A2$.

Proof of the Joint Consistency Theorem, VII

Now we use compactness, and find a countable model $\mathfrak{D}$ of Σ^* with a non-standard element d.

- As in the proof of the Lindström Theorems, we find that in $\mathfrak{D}$ the σ reducts $\mathfrak{D}_0|_\sigma$ and $\mathfrak{D}_1|_\sigma$ of the models $\mathfrak{D}_0$ and $\mathfrak{D}_1$ defined by U_0 and U_1 in $\mathfrak{D}$ are isomorphic.

- To get a model of $\Sigma_o \cup \Sigma_1$ we take the disjoint union of $\mathfrak{D}_0$ and $\mathfrak{D}_1$ and identify the isomorphic parts $\mathfrak{D}_0|_\sigma$ and $\mathfrak{D}_1|_\sigma$.

To extend the proof to uncountable theories we use again compactness.

Q.E.D.

The Robinson (aka Joint Consistency) Property

Let $\mathcal{L}$ be a Lindström Logic.

$\mathcal{L}$ has the Robinson or Joint Consistency Property, ROB, if the following holds:

Let τ_0, τ_1, σ be three countable vocabularies, with $\sigma = \tau_0 \cap \tau_1$.

Let $\Sigma_0 \subseteq \mathcal{L}(\tau_0)$, and $\Sigma_1 \subseteq \mathcal{L}(\tau_1)$ be each satisfiable.

Let $\Sigma \subseteq \mathcal{L}(\sigma)$ a complete (satisfiable) set of sentences, and assume that for each $i = 0, 1$ the theories $\Sigma_i \cup \Sigma$ are satisfiable.

Then $\Sigma_0 \cup \Sigma_1 \cup \Sigma$ is satisfiable.

Joint Consistency implies Compactness

J.A. Makowsky and S. Shelah 1976, D. Mundici 1979



Theorem:

Let $\mathcal{L}$ be a Lindström Logic which satisfies the Joint Consistency Theorem.

Then $\mathcal{L}$ is compact.

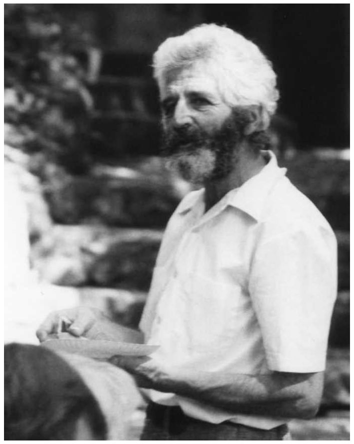
Corollary:

If $\mathcal{L}$ satisfies ROB and LS then $\mathcal{L}$ is of equal expressive power as FOL.

Outline of Part 5.2

- Craig's Interpolation Theorem
- Δ-Interpolation
- Beth's Definability Theorem

Craig's Interpolation Theorem (1957)



William Craig (1918-2016)

UC Berkeley obituary

R. Zach's obituary of W. Craig

Craig's Interpolation Theorem

Let $\bar{R}, \bar{S}, \bar{T}$ be three disjoint finite relational vocabularies.

Let $\phi(\bar{R}, \bar{S})$ and $\psi(\bar{S}, \bar{T})$ two FOL-sentences such that

$$\models \phi(\bar{R}, \bar{S}) \rightarrow \psi(\bar{S}, \bar{T}),$$

then there is a FOLsentence $\theta(\bar{S})$ such that

$$\models \phi(\bar{R}, \bar{S}) \rightarrow \theta(\bar{S}) \text{ and } \models \theta(\bar{S}) \rightarrow \theta(\bar{S}, \bar{T})$$

Equivalently, if

$$\phi(\bar{R}, \bar{S}) \models \psi(\bar{S}, \bar{T}),$$

then there is a FOLsentence $\theta(\bar{S})$ such that

$$\phi(\bar{R}, \bar{S}) \models \theta(\bar{S}) \text{ and } \theta(\bar{S}) \models \theta(\bar{S}, \bar{T})$$

Proof via Joint Consistency (version (ii))

Let $\Sigma_0 = \{\phi(\bar{R}, \bar{S})\}$ and $\Sigma_1 = \{\neg\psi(\bar{S}, \bar{T})\}$.

Assume $\Sigma_0 \cup \Sigma_1$ be unsatisfiable.

So there is $\theta(\bar{S})$ such that $\Sigma_0 \models \theta(\bar{S})$ and $\Sigma_1 \models \neg\theta(\bar{S})$.

In other words

$$\phi(\bar{R}, \bar{S})\} \models \theta(\bar{S})$$

and

$$\neg\psi(\bar{S}, \bar{T}) \models \neg\theta(\bar{S})$$

or, equivalently,

$$\theta(\bar{S}) \models \psi(\bar{S}, \bar{T})$$

Q.E.D.

The case of Second Order Logic SOL is trivial

In case of SOL we can write the hypothesis as

$$\models \forall \bar{R}, \bar{T} \left(\phi(\bar{R}, \bar{S}) \rightarrow \psi(\bar{S}, \bar{T}), \right)$$

But this is equivalent to

$$\models \left(\exists \bar{R} \phi(\bar{R}, \bar{S}) \rightarrow \forall \bar{T} \psi(\bar{S}, \bar{T}), \right)$$

hence, we can take

$$\theta(\bar{S}) = \exists \bar{R} \phi(\bar{R}, \bar{S})$$

because

$$\phi(\bar{R}, \bar{S}) \models \exists \bar{R} \phi(\bar{R}, \bar{S})$$

Q.E.D.

The case of Propositional Logic.

Let $\bar{p}, \bar{q}, \bar{r}$ be different propositional variables.

Let $\phi(\bar{p}, \bar{q})$ and $\psi(\bar{q}, \bar{r})$ two propositional formulas.

Assume

$$\phi(\bar{p}, \bar{q}) \models \psi(\bar{q}, \bar{r}).$$

Then there is a propositional formula $\theta(\bar{q})$ such that

$$\phi(\bar{p}, \bar{q}) \models \theta(\bar{q}) \text{ and } \theta(\bar{q}) \models \theta(\bar{q}, \bar{r})$$

ee this we use

Every quantified propositional formula is logically equivalent to a quantifier-free propositional formula.

We write the hypothesis as

$$\models \forall \bar{p}, \bar{r}\, (\phi(\bar{p}, \bar{q}) \to \psi(\bar{q}, \bar{r}),)$$

and proceed as in the case of SOL.

We put $\theta(\bar{q})$ to be the quantifier-free formula equivalent to $\exists \bar{p} \phi(\bar{p}, \bar{q})$.

Note that $\theta(\bar{q})$ grows exponentially in the size of $\bar{p}$.

Implicite vs explicite definability

Let τ be vocabulary, and let R, S be two relation symbol of arity r which do not occur in τ. Let $\phi(R) \in \mathrm{FOL}(\tau \cup \{R\})$, and $\phi(S)$ the result of substituting S for R in $\phi(R)$.

- $\phi(R)$ defines R implicitly if

$$(\phi(R) \wedge \phi(S)) \models \forall x_1, \ldots, x_r (R(x_1, \ldots, x_r) \leftrightarrow S(x_1, \ldots, x_r))$$

- A formula $\theta(x_1, \ldots, x_r) \in \mathrm{FOL}(\tau)$ defines R explicitly using $\phi(R)$ if

$$\phi(R) \models \forall x_1, \ldots, x_r (R(x_1, \ldots, x_r) \leftrightarrow \theta(x_1, \ldots, x_r))$$

Beth's Definability Theorem (1953)

Theorem:

$\phi(R)$ defines R implicitly iff

there is $\theta(x_1, \ldots, x_r) \in \mathsf{FOL}(\tau)$ which defines R explicitly using $\phi(R)$.

Proof of Beth's Definability Theorem

$\phi(R)$ defines R implicitly:

$$(\phi(R) \wedge \phi(S)) \models \forall x_1, \ldots, x_r (R(x_1, \ldots, x_r) \leftrightarrow S(x_1, \ldots, x_r))$$

Hence we have, for constant symbols $c_1, \ldots, c_r$ not occurring in τ:

$$(\phi(R) \wedge R(c_1, \ldots, c_r) \models \phi(S) \wedge S(c_1, \ldots, c_r) \qquad (*)$$

Let $\theta(c_1, \ldots, c_r)$ be an interpolant for (*).

Then $\theta(x_1, \ldots, x_r)$ is an explicit definition of R for $\phi(R)$.

Q.E.D.

History

- Beth's Definability Theorem was published in 1953:
  E. W. Beth,
  On Padoa's method in the theory of definitions,
  Indagationes Mathematicae, vol. 15 (1953), pp. 330-339.

- Craig's Interpolation Theorem was published in 1957:
  W. Craig, Three uses of the Herbrand-Gentzen theorem in relating model theory and proof theory, The Journal of Symbolic Logic 22 (1957), no. 3, 269-285.

- A. Robinson's Joint Consistence Theorem was published in 1956:
  A. Robinson, A result on consistency and its application to the theory of definitions, Indagationes Mathematicae, vol. 18 (1956), pp. 47-58.

- Lindström's Theorem was published in 1966 and 1969:
  P. Lindström,
  First Order Predicate Logic with Generalized Quantifiers,
  Theoria, 32 (1966), pp. 186-195.
  On extensions of elementary logic, Theoria, 35 (1969), pp. 1-11.

Our proofs are in reverse

- We use Lindström's magic model $\rightarrow$ Joint Consistency
- Joint Consistence $\rightarrow$ Craig's Interpolation
- Craig's Interpolation $\rightarrow$ Beth's Definability Theorem.
- Beth's Theorem originally had an algebraic proof.
- Craig's Theorem was proven via Proof Theory
- A. Robinson used Joint Consistency to prove Beth's Theorem model-theoretically.

G. Kreisel's Theorem (1961)

Georg Kreisel (1923-2015)

Already in 1961 G. Kreisel observed that Craig's Theorem is not constructive in the following sense.

Theorem: G. Kreisel 1961 and H. Friedman 1976

Given two formulas $\phi_1 \in \mathrm{FOL}(\tau_1)$ and $\phi_2 \in \mathrm{FOL}(\tau_2)$ such that $\models \phi_1 \rightarrow \phi_2$ there is NO recursive function which computes an interpolant θ, nor is there a recursive bound on the number of quantifier alternations of the interpolant θ.

A detailed proof was published by H. Friedman
(The complexity of Explicit Definitions, Advances in Mathematics, 1976)

Is interpolation a model theoretic property of logics?

- Craig's Interpolation Theorem has a model theoretic proof.
- But to construct the interpolant, one has to have the

  complete proof sequence

  of $\phi \to \psi$.
- Recent work studies cases where the interpolant can be computed efficiently.

Craig's Theorem for Lindström Logics with LS

Theorem: J. Barwise (1974) and JAM (1973)

- If $\mathcal{L}$ is a logic which satisfies LS and Craig which is more expressive than FOL then it contains the recursive fragment of $\mathcal{L}_{\omega_1,\omega}$ (via the first admissible set containing ω).
- If $\mathcal{L}$ is a logic which satisfies LS and Craig and such that all countable structures are $\aleph_0$-categorical, then $\mathcal{L}$ is at least as expressive as full $\mathcal{L}_{\omega_1,\omega}$.

In general, for Lindström Logics, it is rare to satisfy Craig's Interpolation Theorem.

The re-emergence of Craig's Theorem

In the last 20 years,

Interpolation Theorems

were and are used in

model checking and verification in general.