0-1 Laws for Fragments of Existential Second-Order Logic: A Survey

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Abstract. The probability of a property on the collection of all finite relational structures is the limit as $n \rightarrow \infty$ of the fraction of structures with n elements satisfying the property, provided the limit exists. It is known that the 0-1 law holds for every property expressible in first-order logic, i.e., the probability of every such property exists and is either 0 or 1. Moreover, the associated decision problem for the probabilities is solvable.
In this survey, we consider fragments of existential second-order logic in which we restrict the patterns of first-order quantifiers. We focus on fragments in which the first-order part belongs to a prefix class. We show that the classifications of prefix classes of first-order logic with equality according to the solvability of the finite satisfiability problem and according to the 0-1 law for the corresponding Σ^1_1 fragments are identical, but the classifications are different without equality.

1 Introduction

In recent years a considerable amount of research activity has been devoted to the study of the model theory of finite structures [EF95]. This theory has interesting applications to several other areas including database theory [AHV95] and complexity theory [Imm98]. One particular direction of research has focused on the asymptotic probabilities of properties expressible in different languages and the associated decision problem for the values of the probabilities [Com88].

In general, if C is a class of finite structures over some vocabulary and if P is a property of some structures in C, then *the asymptotic probability* $\mu(P)$ on C is the limit as $n \rightarrow \infty$ of the fraction of the structures in C with n elements which satisfy P, provided that the limit exists. We say that P *is almost surely true* on C in case $\mu(P)$ is equal to 1. Combinatorialists have studied extensively the asymptotic probabilities of interesting properties on the class G of all finite graphs. It is, for example, well known and easy to prove that μ(connectivity)=1, while μ(k-colorabilty)=0, for every $k > 0$ [Bol85]. A theorem of Pósa [Pos76] implies that μ(Hamiltonicity)=1.

Glebskii et al. [GKLT69] and independently Fagin [Fag76] were the first to establish a fascinating connection between logical definability and asymptotic probabilities. More specifically, they showed that if C is the class of all finite structures over some

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⋆⋆ Work partially supported by NSF grant CCR-9700061. Work partly done at LIFO, University of Orléans relational vocabulary and if P is an arbitrary property expressible in first-order logic (with equality), then $\mu(P)$ exists and is either 0 or 1. This result is known as *the 0-1 law for first-order logic*. The proof of the 0-1 law also implies that the decision problem for the value of the probabilities of first-order sentences is solvable. This should be contrasted with Trakhtenbrot's [Tra50] classical theorem to the effect that the set of first-order sentences which are true on *all* finite relational structures is unsolvable, assuming that the vocabulary contains at least one binary relation symbol.

It is well known that first-order logic has very limited expressive power on finite structures (cf. [EF95]). For this reason, one may want to investigate asymptotic probabilities for higher-order logics. Unfortunately, it is easy to see that the 0-1 law fails for second-order logic; for example, parity is definable by an existential second-order sentence. Moreover, the 0-1 laws fails even for existential monadic second-order logic [KS85,Kau87]. In view of this result, it is natural to ask: are there fragments of second order-logic for which a 0-1 law holds?

The simplest and most natural fragments of second-order logic are formed by considering second-order sentences with only existential second-order quantifiers or with only universal second-order quantifiers. These are the well known classes of Σ^1_1 and Π^1_1 sentences respectively. Fagin [Fag74] proved that a property is Σ^1_1 definable if and only if it is NP-computable. As we observed, the 0-1 law fails for Σ^1_1 in general (and consequently for Π^1_1 as well). Moreover, it is not hard to show that the Σ^1_1 sentences having probability 1 form an unsolvable set.

In view of these facts, we concentrate on fragments of Σ^1_1 sentences in which we restrict the pattern of the first-order quantifiers that occur in the sentence. If $\mathcal{F}$ is a class of first-order sentences, then we denote by $\Sigma^1_1(\mathcal{F})$ the class of all Σ^1_1 sentences whose first-order part is in $\mathcal{F}$. Two remarks are in order now. First, if $\mathcal{F}$ is the class of all $\exists^*\forall^*\exists^*$ first-order sentences (that is to say, first-order sentences whose quantifier prefix consists of a string of existential quantifiers, followed by a string of universal quantifiers, followed by a string of existential quantifiers), then $\Sigma^1_1(\mathcal{F})$ has the same expressive power as the full Σ^1_1. In other words, every Σ^1_1 formula is equivalent to one of the form $\exists\mathbf{S}\exists\mathbf{x}\forall\mathbf{y}\exists\mathbf{z}\theta(\mathbf{S},\mathbf{x},\mathbf{y},\mathbf{z})$, where θ is a quantifier-free formula, $\mathbf{S}$ is a sequence of second-order relation variables and $\mathbf{x},\mathbf{y},\mathbf{z}$ are sequences of first-order variables (*Skolem normal form*). Second, if $\phi(\mathbf{S})$ is a first-order sentence without equality over the vocabulary $\mathbf{S}$, then $\mu(\exists\mathbf{S}\phi(\mathbf{S})) = 1$ if and only if $\phi(\mathbf{S})$ is finitely satisfiable. Thus, for every first-order class $\mathcal{F}$, the decision problem for $\Sigma^1_1(\mathcal{F})$ sentences having probability 1 is at least as hard as the finite satisfiability problem for sentences in $\mathcal{F}$. The latter problem is known to be unsolvable [Tra50], even in the case where $\mathcal{F}$ is the class of $\exists^*\forall^*\exists^*$ sentences ([BGG97]). As a result, in order to pursue positive solvability results one has to consider fragments $\Sigma^1_1(\mathcal{F})$, where $\mathcal{F}$ is a class for which the finite satisfiability problem is solvable. Such classes $\mathcal{F}$ of first-order sentences are said to be *docile* [DG79].

In first-order logic *without equality*, there are three docile *prefix classes*, i.e., classes of first-order sentences defined by their quantifier prefix [BGG97]:

– The *Bernays-Schönfinkel class*, which is the collection of all first-order sentences with prefixes of the form $\exists^*\forall^*$ (i.e., the existential quantifiers precede the universal quantifiers).

– The *Ackermann class*, which is the collection of all first-order sentences with prefixes of the form $\exists^*\forall\exists^*$ (i.e., the prefix contains a single universal quantifier).
– The *Gödel class*, which is the collection of all first-order sentences with prefixes of the form $\exists^*\forall\forall\exists^*$ (i.e., the prefix contains two consecutive universal quantifiers).

These three classes are also the only prefix classes that have a solvable satisfiability problem [BGG97]. In first-order logic *with equality*, the Gödel class is not docile and its satisfiability problem is not solvable [Gol84]. This is the only class where equality makes a difference.

We focus here on the question whether the 0-1 law holds for the Σ_1^1 fragments defined by first-order prefix classes, and whether or not the associated decision problem for the probabilities is solvable. This can be viewed as a classification of the prefix classes according to whether the corresponding Σ_1^1 fragments have a 0-1 law. This classification project was launched in [KV87] and was completed only recently in [LeB98]. For first-order logic with equality, the classifications of prefix classes according to their docility, i.e., according to the solvability of their finite satisfiability problem, and according to the 0-1 law for the corresponding Σ_1^1 fragment are identical. Moreover, 0-1 laws in this classification are always accompanied by solvability of the decision problem for the probabilities. This is manifested by the positive results for the classes Σ_1^1(Bernays-Schönfinkel) and Σ_1^1(Ackermann), and the negative results for the other classes. For first-order logic with equality, the two classification differ, as the 0-1 law fails for the class Σ_1^1(Gödel) and the association classification problem is undecidable.

This paper is a survey that focuses on the overall picture rather than on technical details. The interested reader is referred to the cited papers for further details. Our main focus here is on positive results involving 0-1 laws. For a survey that focus on negative results, see [LeB00]. For an earlier overview, which includes a focus on expressiveness issues, see [KV89]. See [Lac97] for results on 0-1 laws for second-order fragments that involves alternation of second-order quantifiers.

2 Random Structures

Let **R** be a vocabulary consisting of relation symbols *only* and let C be the collection of all finite relational structures over **R** whose universes are initial segments $\{1, 2, \ldots, n\}$ of the integers. If P is a property of (some) structures in C, then let $\mu_n(P)$ be the fraction of structures in C of cardinality n satisfying P. The *asymptotic probabilty* $\mu(P)$ on C is defined to be $\mu(P) = \lim_{n\to\infty} \mu_n(P)$, provided this limit exists. In this probability space all structures in C with the same number of elements carry the same probability. An equivalent description of this space can be obtained by assigning truth values to tuples independently and with the same probability (cf. [Bol85]).

If L is a logic, we say that the 0-1 *law holds for L on C* in case $\mu(P)$ exists and is equal to 0 or 1 for every property P expressible in the logic L. We write $\Theta(L)$ for the collection of all sentences P in L with $\mu(P) = 1$. Notice that if L is first-order logic, then the existence of the 0-1 law is equivalent to stating that $\Theta(L)$ is a complete theory.

A standard method for establishing 0-1 laws, originating in Fagin [Fag76], is to prove that the following *transfer theorem* holds: there is an infinite structure $\mathbf{A}$ over the vocabulary **R** such that for every property P expressible in L we have: $\mathbf{A} \models P \iff$

$\mu(P) = 1$. It turns out that there is a unique (up to isomorphism) countable structure $\mathbf{A}$ that satisfies the above equivalence for first-order logic and for the fragments of second-order logic considered here. We call $\mathbf{A}$ the *countable random structure over the vocabulary* $\mathbf{R}$. The structure $\mathbf{A}$ is characterized by an infinite set of *extension axioms*, which, intuitively, assert that every $type$ can be $extended$ to every other possible type. More precisely, if $\mathbf{x} = (x_1, \ldots, x_n)$ is a sequence of variables, then a n-$\mathbf{R}$-*type* $t(\mathbf{x})$ *in the variables* $\mathbf{x}$ *over* $\mathbf{R}$ is a maximal consistent set of equality and negated equality formulas and atomic and negated atomic formulas from the vocabulary $\mathbf{R}$ in the variables $x_1, \ldots, x_n$. We say that a $(n+1)$-$\mathbf{R}$-*type* $s(\mathbf{x}, z)$ *extends the type* $t(\mathbf{x})$ if t is a subset of s. Every type $t(\mathbf{x})$ can be also viewed as a quantifier-free formula that is the conjunction of all members of $t(\mathbf{x})$. With each pair of types s and t such that s extends t we associate a first-order *extension axiom* τ which states that $(\forall \mathbf{x})(t(\mathbf{x}) \rightarrow (\exists z)s(\mathbf{x}, z))$.

Let T be the set of all extension axioms. The theory T was studied by Gaifman [Gai64], who showed, using a *back and forth* argument, that every two countable models of T are isomorphic (i.e., *T is an ω-categorical theory*). The extension axioms can also be used to show that the unique (up to isomorphism) countable model $\mathbf{A}$ of T is *universal* for all countable structures over $\mathbf{R}$, i.e., if $\mathbf{B}$ is a countable structure over $\mathbf{R}$, then there is a substructure of $\mathbf{A}$ that is isomorhic to $\mathbf{B}$.

Fagin [Fag76] realized that the extension axioms are relevant to the study of probabilities on finite structures and proved that on the class C of all finite structures over a vocabulary $\mathbf{R}$ $\mu(\tau) = 1$ for every extension axiom τ. The 0-1 law for first-order logic and the transfer theorem between truth of first-order sentences on $\mathbf{A}$ and almost sure truth of such sentences on C follows from these results by a compactness argument. We should point out that there are different proofs of the 0-1 law for first-order logic, which have a more elementary character (cf. [GKLT69,Com88]). These proofs do not deploy infinite structures or the compactness theorem and they bypass the transfer theorem. In contrast, the proofs of the 0-1 laws for fragments of second-order logic that we present here do involve infinitistic methods. Lacoste showed how these infinitistic arguments can be avoided [Lac96].

Since the set T of extension axioms is recursive, it also follows that $\Theta(L)$ is recursive, where L is first-order logic. In other words, there is an algorithm to decide the value (0 or 1) of the asymptotic probability of every first-order sentence. The computational complexity of this decision problem was investigated by Grandjean [Gra83], who showed that it is PSPACE-complete, when the underlying vocabulary $\mathbf{R}$ is assumed to be *bounded* (i.e., there is a some bound on the arity of the relation symbols in σ).

3 Existential and Universal Second-Order Sentences

The Σ^1_1 and Π^1_1 formulas form the syntactically simplest fragment of second-order logic. A Σ^1_1 formula over a vocabulary $\mathbf{R}$ is an expression of the form $(\exists \mathbf{S})\theta(\mathbf{S})$, where $\mathbf{S}$ is a sequence of relation symbols not in the vocabulary $\mathbf{R}$ and $\theta(\mathbf{S})$ is a first-order formula over the vocabulary $\mathbf{R} \cup \mathbf{S}$. A Π^1_1 formula is an expression of the form $(\forall \mathbf{S})\theta(\mathbf{S})$, where $\mathbf{S}$ and $\theta(\mathbf{S})$ are as above.

Both the 0-1 law and the transfer theorem fail for arbitrary Σ^1_1 and Π^1_1 sentences. Consider, for example, the statement “there is relation that is the graph of a permutation in which every element is of order 2". On finite structures this statement is true exactly when the universe of the structure has an even number of elements and, as a result, it has no asymptotic probability. This statement, however, is expressible by a Σ^1_1 sentence, which, moreover, is true on the countable random structure $\mathbf{A}$. Similarly, the statement "there is a total order with no maximum element" is true on the countable random structure $\mathbf{A}$, but is false on every finite structure. Notice that in the two preceding examples the transfer theorem for Σ^1_1 sentences fails in the direction from truth on the countable random structure $\mathbf{A}$ to almost sure truth on finite structures. In contrast, the following simple lemma shows that this direction of the transfer theorem holds for all Π^1_1 sentences.

Lemma 1. [KV87] *Let $\mathbf{A}$ be the countable random structure over $\mathbf{R}$ and let $(\forall \mathbf{S})\theta(\mathbf{S})$ be an arbitrary Π^1_1 sentence. If $\mathbf{A} \models (\forall \mathbf{S})\theta(\mathbf{S})$, then there is a first order sentence ψ over the vocabulary σ such that: $\mu(\psi) = 1$ and $\models \psi \rightarrow (\forall \mathbf{S})\theta(\mathbf{S})$. In particular, every Π^1_1 sentence that is true on $\mathbf{A}$ has probability 1 on C.*

The proof of Lemma 1 uses the Compactness Theorem. For an approach that avoid the usage of infinitistic arguments, see [Lac96].

Corollary 1. [KV87] *Every Σ^1_1 sentence that is false on the countable random structure $\mathbf{A}$ has probability 0 on C.*

Corollary 2. [KV87] *The set of Π^1_1 sentences that are true on $\mathbf{A}$ is recursively enumerable.*

Proof: It shown in [KV87] that $\mathbf{A} \models (\forall \mathbf{S})\theta(\mathbf{S})$ iff $(\forall \mathbf{S})\theta(\mathbf{S})$ is logically implied by the set T of extension axioms. ∎

We investigate here classes of Σ^1_1 and Π^1_1 sentences that are obtained by restricting appropriately the pattern of the first-order quantifiers in such sentences. If $\mathcal{F}$ is a class of first-order formulas, then we write $\Sigma^1_1(\mathcal{F})$ for the collection of all Σ^1_1 sentences whose first-order part is in $\mathcal{F}$.

The discussion in the introduction suggests that we consider prefix classes $\mathcal{F}$ that are docile, i.e., they have a solvable finite satisfiability problem. Thus, we focus on the following classes of existential second-order sentences:

- The class $\Sigma^1_1(\exists^*\forall^*)$ of Σ^1_1 sentences whose first-order part is a *Bernays-Schönfinkel* formula.
- The class $\Sigma^1_1(\exists^*\forall\exists^*)$ of Σ^1_1 sentences whose first-order part is an *Ackermann* formula.
- The class $\Sigma^1_1(\exists^*\forall\forall\exists^*)$ of Σ^1_1 sentences whose first-order part is a *Gödel* formula.

We also refer to the above as the Σ^1_1(Bernays-Schönfinkel) class, the Σ^1_1(Ackermann) class, and the Σ^1_1(Gödel) class, respectively. We consider these classes both with and without equality.

Fagin [Fag74] showed that a class of finite structures over a vocabulary $\mathbf{R}$ is NP computable if and only if it is definable by a Σ^1_1 sentence over $\mathbf{R}$. The restricted classes of Σ^1_1 sentences introduced above can not express all NP problems on finite structures. In spite of their syntactic simplicity, however, the classes $\Sigma^1_1(\exists^*\forall^*)$, $\Sigma^1_1(\exists^*\forall\exists^*)$ and $\Sigma^1_1(\exists^*\forall\forall\exists^*)$ can express natural NP-complete problems [KV87,KV90].

4 The Class Σ_1^1(Bernays-Schönfinkel) with Equality

4.1 0-1 Law

Lemma 1 and Corollary 1 reveal that in order to establish the 0-1 law for a class $\mathcal{F}$ of existential second-order sentences it is enough to show that if Ψ is a sentence in $\mathcal{F}$ that is true on the countable random structure $\mathbf{A}$, then $\mu(\Psi) = 1$. In this section we prove this to be the case for the class of Σ_1^1(Bernays-Schönfinkel) sentences.

Lemma 2. [KV87] *Let $(\exists \mathbf{S})(\exists \mathbf{x})(\forall \mathbf{y})\theta(\mathbf{S}, \mathbf{x}, \mathbf{y})$ be a $\Sigma_1^1(\exists^*\forall^*)$ sentence that is true on the countable random structure $\mathbf{A}$. Then there is a first order sentence ψ over σ such that $\mu(\psi) = 1$ and $\models_{fin} \psi \rightarrow (\exists \mathbf{S})(\exists \mathbf{x})(\forall \mathbf{y})\theta(\mathbf{S}, \mathbf{x}, \mathbf{y})$, where $\models_{fin}$ denotes truth in all finite structures. In particular, if Ψ is a $\Sigma_1^1(\exists^*\forall^*)$ sentence that is true on $\mathbf{A}$, then $\mu(\Psi) = 1$.*

Proof: Let $\mathbf{a} = (a_1, ..., a_n)$ be a sequence of elements of $\mathbf{A}$ that witness the first-order existential quantifiers $\mathbf{x}$ in $\mathbf{A}$. Let $\mathbf{A_0}$ be the finite substructure of $\mathbf{A}$ with universe $\{a_1, ..., a_n\}$. Then there is a first-order sentence ψ, which is the conjunction of a finite number of the extension axioms, having the property that every model of it contains a substructure isomorphic to $\mathbf{A_0}$. Now assume that $\mathbf{B}$ is a finite model of ψ. Using the extension axioms we can find a substructure $\mathbf{B}^*$ of the random structure $\mathbf{A}$ that contains $\mathbf{A_0}$ and is isomorphic to $\mathbf{B}$. Since universal statements are preserved under substructures, we conclude that $\mathbf{B} \models (\exists \mathbf{S})(\exists \mathbf{x})(\forall \mathbf{y})\theta(\mathbf{S}, \mathbf{x}, \mathbf{y})$, where $\mathbf{x}$ is interpreted by $\mathbf{a}$ and $\mathbf{S}$ is interpreted by the restriction to $\mathbf{B}$ of the relations on $\mathbf{A}$ that witness the existential second-order quantifiers. ∎

From Lemmas 1 and 2 we infer immediately the 0-1 law and the transfer theorem for the class $\Sigma_1^1(\exists^*\forall^*)$.

Theorem 1. [KV87] *The 0-1 law holds for Σ_1^1(Bernays-Schönfinkel) sentences on the class C of all finite structures over a relational vocabulary $\mathbf{R}$. Moreover, if $\mathbf{A}$ is the countable random structure and Ψ is a Σ_1^1(Bernays-Schönfinkel) sentence, then $\mathbf{A} \models \Psi \Longleftrightarrow \mu(\Psi) = 1$.*

4.2 Solvability

As mentioned in Section 2, the proof of the 0-1 law for first-order logic showed also the solvability of the decision problem for the values (0 or 1) of the probabilities of first-order sentences. The preceding proof of the 0-1 law for Σ_1^1(Bernays-Schönfinkel) sentences does not yield a similar result for the associated decision problem for the probabilities of such sentences. Indeed, the only information one can extract from the proof is that the Σ_1^1(Bernays-Schönfinkel) sentences of probability 0 form a recursively enumerable set. We now show that the decision problem for the probabilities of sentences in the class Σ_1^1(Bernays-Schönfinkel) is solvable. We do this by proving that satisfiability of such sentences on $\mathbf{A}$ is equivalent to the existence of certain *canonical* models. For simplicity we present the argument for $\Sigma_1^1(\forall^*)$ sentences, i.e., sentences of the form

$$\exists S_1 ... \exists S_l \forall y_1 ... \forall y_m \theta(S_1, ..., S_l, y_1, ..., y_m).$$

Assume that the vocabulary σ consists of a sequence $\mathbf{R} = \langle R_i, i \in I\rangle$ of relation variables R_i. If B is a set and, for each $i \in I$, R_i^B is a relation on B of the same arity as that of R_i, then we write $\mathbf{R}^B$ for the sequence $\langle R_i^B, i \in I\rangle$. Let $<$ be a new binary relation symbol and consider structures $\mathbf{B} = (B, \mathbf{R}^B, <^B)$ in which $<^B$ is a *total ordering*. Let *k* be a positive integer. We say that $\mathbf{B}$ is *k-rich* if for every structure $\mathbf{D}$ with *k* elements over $\mathbf{R} \cup \{<\}$ (where $<$ is interpreted by a total ordering) there is a substructure $\mathbf{B}^*$ of $\mathbf{B}$ that is isomorphic to $\mathbf{D}$. Notice that the isomorphism takes into account *both* the relations R_i^D and the total ordering $<^D$.

Assume that S^B is an n-ary relation on B. We say that S^B is *canonical for the structure* $\mathbf{B} = (B, \mathbf{R}^B, <^B)$ if for every sequence $\mathbf{b} = (b_1, ..., b_n)$ from B the truth value of $S^B(b_1, ..., b_n)$ depends *only* on the isomorphism type of the substructure of $\mathbf{B}$ with universe $\{b_1, ..., b_n\}$. An expanded structure $\mathbf{B}^* = (B, \mathbf{R}^B, <^B, S_1^B, ..., S_l^B)$ is *canonical* if every relation S_i^B, $1 \leq i \leq l$, is canonical on $\mathbf{B}$. The intuition behind canonical structures is that the relations S_i^B are determined completely by the $(\mathbf{R}, <)$-types.

We can state now the main technical result of this section.

Theorem 2. [KV87] *Let* $\mathbf{A}$ *be the countable random structure over the vocabulary* σ *and let* Ψ *be a* Σ_1^1 *sentence of the form* $\exists S_1 ... \exists S_l \forall y_1 ... \forall y_m \theta(S_1, ..., S_l, y_1, ..., y_m)$. *Then the following are equivalent:*

1. $\mathbf{A} \models \Psi$.
2. *There is a finite canonical structure* $\mathbf{B}^* = (B, \mathbf{R}^B, <^B, S_1^B, ..., S_l^B)$ *which is k-rich for every k*$\leq m$ *and such that* $\mathbf{B}^* \models \forall y_1 ... \forall y_m \theta(S_1, ..., S_l, y_1, ..., y_m)$.

In showing that $(1) \implies (2)$ we will use certain Ramsey-type theorems that were proved by [NR77,NR83] and independently by [AH78]. We follow here the notation and terminology of [AH78] in stating these combinatorial results.

If X is a set and j is an integer, then $[X]^j$ is the collection of all subsets of X with j elements and $[X]^{\leq n} = \bigcup_{j \leq n} [X]^j$. A *system of colors* is a sequence $\mathbf{K} = (K_1, ..., K_n)$ of finite nonempty sets. A $\mathbf{K}$*-colored set* consists of a finite set X, a (total) ordering $<^X$ on X and a function $f : [X]^{\leq n} \mapsto K_1 \cup ... \cup K_n$ such that $f(Z) \in K_j$ for every $Z \in [X]^j$ and $j \leq n$.

It is clear that every $\mathbf{K}$-colored set is isomorphic to a unique $\mathbf{K}$-*pattern*, that is a $\mathbf{K}$-colored set whose underlying set is an integer. If e, M are integers, $\mathbf{K}$ is a system of colors, P, Q are $\mathbf{K}$-patterns, then $Q \hookrightarrow (P)^{\leq e}_M$ means that for every $\mathbf{K}$-colored set (X, f) of pattern Q and every partition $F : (X)^{\leq e} \mapsto M$ there is a subset Y of X such that $(Y, f \mid_Y)$ is of pattern P, and Y is *conditionally monochromatic* for F as $\mathbf{K}$-colored set, i.e. for $Z \in [Y]^j$ the value $F(Z)$ depends *only* on the $\mathbf{K}$-pattern of $(Z, f \mid_Z)$.

By iterated applications of Theorem 2.2 in [AH78], we can derive the following generalization of the classical Ramsey theorem [Ram28]:

Theorem 3. [KV87] *For arbitrary integers* e, M*, a system of colors* $\mathbf{K}$*, and a* $\mathbf{K}$*-pattern* P*, there is a* $\mathbf{K}$*-pattern* Q *such that* $Q \hookrightarrow (P)^{\leq e}_M$.

With every finite vocabulary $\mathbf{R}$ in which the maximum arity is n we can associate a system of colors $\mathbf{K}$ such that every finite structure $\mathbf{B} = (B, \mathbf{R}^B, <^B)$, where $<^B$ is a total ordering on B, can be *coded* by a $\mathbf{K}$-colored set $(B, <^B, f)$ with $f : [B]^{\leq n} \mapsto$

$K_1 \cup ... \cup K_n$. For example, if σ consists of a single binary relation R, then K_1 has 2 elements, K_2 has 4 elements, and $f : [B]^{\leq 2} \mapsto K_1 \cup K_2$ is such that the value $f(\{x\})$ depends only on the truth value of $R^B(x,x)$, while the value $f(\{x,y\})$ depends only on the truth values of $R^B(min(x,y), max(x,y))$ and $R^B(max(x,y), min(x,y))$. Conversely, from every such **K**-pattern we can *decode* a finite structure **B**.

We now have all the combinatorial machinery needed to outline the ideas in the proof of Theorem 2.

Sketch of Proof of Theorem 2.

$(1) \implies (2)$ Let Ψ be the Σ_1^1 sentence Ψ such that $\mathbf{A} \models \Psi$ and assume for simplicity that Ψ has only one ternary second-order existential variable S. We use the ternary relation S^A witnessing S on **A** to partition $[A]^{\leq 3}$ according to the *pure* S-type of a set $Z \in [A]^{\leq 3}$. This means that A is partitioned into two pieces defined by the truth value of $S^A(x,x,x)$, $[A]^2$ into $2^{2^3-2} = 64$ pieces defined by the truth values of $S^A(min(x,y), max(x,y), max(x,y))$, etc., and finally $[A]^3$ is partitioned into $2^{3!} = 64$ pieces defined by the truth values of $S^A(min(x,y,z), mid(x,y,z), max(x,y,z))$, etc..

Let $\mathbf{B} = (B, \mathbf{R}^B, <^B)$ be a finite structure which is k-rich for every $k \leq m$ and let P be a **K**-pattern which codes **B** in the way described above. We apply now Theorem 3 for $e = 3$, $M = 2 + 64 + 64 = 130$, and for P coding **B**. Let Q be a **K**-pattern such that $Q \hookrightarrow (P)^{\leq 3}_M$ and let **C** be the structure coded by Q. Using the extension axioms for the countable random structure **A** we can find in **A** a substructure $\mathbf{C_1}$ isomorphic to **C**. But now Theorem 3 guarantees that $\mathbf{C_1}$ contains a substructure $\mathbf{B_1}$ which is isomorphic to **B** and is conditionally monochromatic as a **K**-colored pattern. The structure $\mathbf{B_1}$ is k-rich for every $k \leq m$ and by taking the restriction of S^A on B_1 we can expand $\mathbf{B_1}$ to a canonical model $\mathbf{B}^*$ of $\forall y_1 ... \forall y_m \theta(S, y_1, ..., y_m)$, since universal sentences are preserved under substructures.

$(2) \implies (1)$ From every canonical, k-rich model ($1 \leq k \leq m$) of

$$\forall y_1 ... \forall y_m \theta(S, y_1, ..., y_m)$$

we can build a relation S^A witnessing the second order existential quantifier in Ψ by assigning tuples to S^A according to their $(\mathbf{R}, <)$-type. ∎

Theorem 2 implies that the set of $\Sigma_1^1(\exists^* \forall^*)$ properties having probability 1 is recursively enumerable. On the other hand, Theorem 1 and Corollary 2 together imply that the complement of this set is also recursively enumerable. Thus, we have established that the decision problem for the probabilities of strict Σ_1^1 properties is solvable. This proof does not give, however, any complexity bounds for the problem. In [KV87] we analyzed the computational complexity of this decision problem and showed that it is NEXPTIME-complete for bounded vocabularies and 2NEXPTIME for unbounded vocabularies.

5 The Class Σ_1^1(Ackermann) with Equality

Our goal here is to establish the following:

Theorem 4. [KV90] *Let* $\mathbf{A}$ *be the countable random structure over the vocabulary* $\mathbf{R}$ *and let* Ψ *be a* Σ^1_1*(Ackermann) sentence. If* $\mathbf{A} \models \Psi$*, then* $\mu(\Psi) = 1$.

This theorem will be obtained by combining three separate lemmas. Since the whole argument is rather involved, we start with a "high-level" description of the structure of the proof.

We first isolate a syntactic condition (condition (χ) below) for Σ^1_1(Ackermann) sentences and in Lemma 3 we show that if Ψ is a Σ^1_1(Ackermann) sentence which is true on $\mathbf{A}$, then condition (χ) holds for Ψ. At the end, it will actually turn out that this condition (χ) is also sufficient for truth of Σ^1_1(Ackermann) sentences on the countable random structure $\mathbf{A}$. In Lemma 4, we isolate a "richness" property E_s, $s \geq 1$, of (some) finite structures over $\mathbf{R}$ and show that $\mu(E_s) = 1$ for every $s \geq 1$. The proof of this lemma requires certain asymptotic estimates from probability theory, due to Chernoff [Che52]. Finally, in Lemma 5, we prove that if Ψ is a Σ^1_1(Ackermann) sentence for which condition (χ) holds, then for appropriately chosen s and for all large n the sentence Ψ is true on all finite structures of cardinality n over $\mathbf{R}$ that possess property E_s; consequently, $\mu(\Psi) = 1$. In this last lemma, the existence of the predicates $\mathbf{S}$ that witness Ψ is proved by a probabilistic argument, which in spirit is analogous to the technique used by Gurevich and Shelah [GS83] for showing the finite satisfiability property of first-order formulas in the Gödel class without equality.

Let $\mathbf{T}$ be a vocabulary, i.e. a set of relational symbols. Recall that, a k-$\mathbf{T}$-*type* $t(x_1, ..., x_k)$ is a maximal consistent set of equality, negated equality formulas, atomic and negated atomic formulas whose variables are among $x_1, ..., x_k$.

- If $t(x_1, ..., x_k)$ is a k-$\mathbf{T}$-type, then, for every m with $1 \leq m \leq k$, let $t(x_{i_1}, ..., x_{i_m})$ be the m-$\mathbf{T}$-type obtained by deleting from $t(x_1, ..., x_k)$ all formulas in which a variable $y \neq x_{i_1}, \ldots, x_{i_m}$ occurs.
- If $\mathbf{S} \subseteq \mathbf{T}$, then the *restriction* of t to $\mathbf{S}$ is the k-$\mathbf{S}$-type obtained by deleting from t all formulas in which a predicate symbols in $\mathbf{T} - \mathbf{S}$ occurs.
- If $t(x_1, ..., x_k, x_{k+1})$ is a $(k+1)$-$\mathbf{T}$-type, and y is a variable different from all the x_i's, then $t(x_1, ..., x_k, x_{k+1}/y)$ is a $(k+1)$-$\mathbf{T}$-type obtained by replacing all occurrences of x_{k+1} by y.
- Let $t(x_1, ..., x_k)$ be a k-$\mathbf{T}$-type, and let $\phi(x_1, ..., x_k)$ be a quantifier-free formula in the variables $x_1, ..., x_k$. We say that t *satisfies* ϕ if ϕ is true under the truth assignment that assigns true to an atomic formula precisely when it is a member of t.

Let Ψ be a Σ^1_1(Ackermann) sentence of the form

$$(\exists \mathbf{S})(\exists x_1) \ldots (\exists x_k)(\forall y)(\exists z_1) \ldots (\exists z_l)\phi(x_1, \ldots, x_k, y, z_1, \ldots, z_l, \mathbf{R}, \mathbf{S}),$$

where ϕ is a quantifier-free formula over the vocabulary ($\mathbf{R}$,$\mathbf{S}$)=$\mathbf{R}\cup\mathbf{S}$.

We say that *condition* (χ) *holds for* Ψ if there is k-($\mathbf{R}$,$\mathbf{S}$)-type $t_0(x_1, ..., x_k)$ and a set P of $(k+1)$-($\mathbf{R}$,$\mathbf{S}$)-types $t(x_1, ..., x_k, y)$ extending $t_0(x_1, ..., x_k)$ such that the following are true:

1. P contains as a member the $(k+1)$-$(\mathbf{R}, \mathbf{S})$-type $t_0^{x_i}(x_1, \ldots, x_k, y)$, for every $i = 1 \ldots k$. Equivalently, for every i, $1 \leq i \leq k$, there is a type $t_i(x_1, \ldots, x_k, y)$ in P such that $t_i(x_1, \ldots, x_k, y/x_i) = t_0(x_1, \ldots, x_k)$.

2. P is **R**-*rich over* $t_0(x_1, ..., x_k)$, i.e., every $(k+1)$-**R**-type $t(x_1, ..., x_k, y)$ extending the restriction of $t_0(x_1, ..., x_k)$ to **R** is itself the restriction of some $(k+1)$-(**R**,**S**)-type in P to **R**.
3. For each $t(x_1, ..., x_k, y)$ in P there is a $(k+l+1)$-(**R**,**S**)-type

$$t'(x_1, \ldots, x_k, y, z_1, \ldots, z_l)$$

such that $t \subseteq t'$, t' satisfies $\phi(x_1, \ldots, x_k, y, z_1, \ldots, z_l)$, and for each z_i, $1 \leq i \leq l$, the $(k+1)$-(**R**,**S**)-type $t'(x_1, ..., x_k, z_i/y)$ is in P.

Lemma 3. [KV90] *Let* **A** *be the countable random structure over the vocabulary* **R** *and let* Ψ *be a* Σ^1_1*(Ackermann) sentence. If* $\mathbf{A} \models \Psi$*, then condition* (χ) *holds for* Ψ.

Proof: (Hint) The type t_0 and the set of types P required in condition (χ) are obtained from the relations on **A** and the elements of **A** that witness the existential second-order quantifiers $(\exists \mathbf{S})$ and the existential first-order quantifiers $(\exists x_1) \ldots (\exists x_k)$ in Ψ respectively. To show that P is **R**-rich, we use the fact that the countable random structure **A** satisfies the extension axioms, which in turn imply that the elements of **A** realize all possible **R**-types. ▮

Let D be a structure over **R** and let $\overline{c} = (c_1, \ldots, c_m)$ be a sequence of elements from D. The *type* $t_{\overline{c}}$ of $\overline{c}$ on D is the unique m-**R**-type $t(z_1, \ldots, z_m)$ determined by the atomic and negated atomic formulas that the sequence $\overline{c}$ satisfies on D, under the assignment $z_i \rightarrow c_i$, $1 \leq i \leq m$. We say that a sequence $\overline{c}$ *realizes* a type t on a structure D if $t_{\overline{c}} = t$.

Let $s \geq 1$ be fixed. We say that a finite structure D over **R** with n elements *satisfies property* E_s if the following holds:

– For every number m with $1 \leq m \leq s$, every sequence $\overline{c} = (c_1, \ldots, c_m)$ from D and every $(m+1)$-**R**-type $t(z_1, \ldots, z_m, z_{m+1})$ extending the type $t_{\overline{c}}$ of $\overline{c}$ on D, there are at least $\sqrt{n}$ different elements d in D such that each sequence $(c_1, \ldots, c_m, d)$ realizes the type $t(z_1, \ldots, z_m, z_{m+1})$.

Lemma 4. [KV90] *For every* $s \geq 1$ *there is a positive constant* c *and a natural number* n_0 *such that for every* $n \geq n_0$ $\mu_n(E_s) \geq 1 - n^{s+1}e^{-cn}$*. In particular,* $\mu(E_s) = 1$*, i.e. almost all structures over* **R** *satisfy property* E_s*, for every* $s \geq 1$.

Proof: (Sketch) The proof of this lemma uses an asymptotic bound on the probability in the tail of the binomial distribution, due to Chernoff [Che52] (cf. also [Bol85]). We first fix a sequence $\overline{c}$ from D and a type t that extends $t_{\overline{c}}$, and apply this bound to the binomial distribution obtained by counting the number of elements d such that the sequence $(c_1, \ldots, c_m, d)$ realizes t. We then iterate through all types and all sequences $\overline{c} = (c_1, \ldots, c_m)$ for $1 \leq m \leq s$. ▮

The last lemma in this section provides the link between condition (χ), property E_s, $s \geq 1$, and satisfiability of Σ^1_1(Ackermann) sentences on finite structures over **R**.

Lemma 5. [KV90] *Let* Ψ *be a* Σ^1_1*(Ackermann) sentence of the form*

$$(\exists \mathbf{S})(\exists x_1) \ldots (\exists x_k)(\forall y)(\exists z_1) \ldots (\exists z_l)\phi(x_1, \ldots, x_k, y, z_1, \ldots, z_l, \mathbf{R}, \mathbf{S})$$

for which condition (χ) *holds. There is a natural number* n_1 *such that for every* $n \geq n_1$*, if* D *is a finite structure over* $\mathbf{R}$ *with* n *elements satisfying property* E_{k+l+1}*, then* $D \models \Psi$.

Proof: (Sketch) The existence of the relations on D that witness the second-order quantifiers $(\exists \mathbf{S})$ in Ψ is proved with a probabilistic argument similar to the one employed by Gurevich and Shelah [GS83] for the finite satisfiability property of the Gödel class without equality. We use condition (χ) to impose on D a probability space of **S** predicates. The richness property E_{k+l+1} is then used to show that with nonzero probability (in this new space) the expansion of D with these predicates satisfies the sentence $(\exists x_1) \ldots (\exists x_k)(\forall y)(\exists z_l) \ldots (\exists z_l)\phi(y, z_1, \ldots, z_l, \mathbf{S})$. ∎

This completes the outline of the proof of Theorem 4. Combining now this theorem with Lemma 1 we derive the main result of this section.

Theorem 5. [KV90] *The 0-1 law holds for the* Σ_1^1(*Ackermann*) *class on the collection* C *of all finite structures over a vocabulary* $\mathbf{R}$. *Moreover, if* $\mathbf{A}$ *is the countable random structure over* $\mathbf{R}$ *and* Ψ *is a* Σ_1^1(*Ackermann*) *sentence, then* $\mathbf{A} \models \Psi \iff \mu(\Psi) = 1$.

Notice that the preceding results also show that a Σ_1^1(Ackermann) sentence Ψ has probability 1 if and only if condition (χ) holds for Ψ. Since condition (χ) is clearly effective, it follows that the decision problem for the values of the probabilities of Σ_1^1(Ackermann) sentences is solvable. In [KV90] we analyzed the computational complexity of this decision problem and showed that it is NEXPTIME-complete for bounded vocabularies and Σ_2^{exp}-complete[1] for unbounded vocabularies.

6 Negative Results and Classifications

The Bernays-Schönfinkel and Ackermann classes are the only docile prefix classes with equality, i.e., they are the only prefix classes of first-order logic with equality for which the finite satisfiability problem is solvable [BGG97]. A key role in this classification was played by the *Gödel class* with equality, which is the class of first-order sentences with equality and with prefix of the form $\forall\forall\exists^*$. In fact, the classification was completed only when Goldfarb [Gol84] showed that the *minimal Gödel class*, i.e., the class of first-order sentences with equality and with prefix of the form $\forall\forall\exists$, is not docile (contradicting an unproven claim in [God32]). We now show that in the presence of equality the same classification holds for the 0-1 law, namely, the 0-1 law holds for the Σ_1^1 fragments that correspond to docile prefix classes.

It is easy to see that the 0-1 law does not hold for the Σ_1^1 fragments that correspond to the prefix classes $\forall\exists\forall$ and $\forall^3\exists$. For example, the PARITY property, i.e., the property "there is an even number of elements" can be expressed by the following $\Sigma_1^1(\forall\exists\forall)$ sentence asserting that "there is a permutation in which every element is of order 2":

$$(\exists S)(\forall x)(\exists y)(\forall z)[S(x,y) \wedge (S(x,z) \to y = z) \wedge (S(x,z) \leftrightarrow S(z,x)) \wedge \neg S(x,x)].$$

[1] Σ_2^{exp} is the second-level of the exponential hierarchy. It can be described as the class of languages accepted by alternating exponential-time Turing machines in two alternations where the machine start state is existential [CKS81] or as the class $\mathrm{NEXP}^{\mathrm{NP}}$ of languages accepted by nondeterministic exponential-time Turing machines with oracles from NP [HIS85].

The statement "there is a permutation in which every element is of order 2" can also be expressed by the following $\Sigma_1^1(\forall\forall\forall\exists)$ sentence

$$(\exists S)(\forall x)(\forall y)(\forall z)(\exists w)[S(x,w) \wedge (S(x,y) \wedge S(x,z) \rightarrow y = z) \wedge (S(x,z) \leftrightarrow S(z,x)) \wedge \neg S(x,x)].$$

Dealing with the class Σ_1^1(Gödel), i.e., the class $\Sigma_1^1(\forall\forall\exists^*)$ is much harder.

Theorem 6. [PS91,PS93] *The 0-1 law fails for the class* $\Sigma_1^1(\forall\forall\exists)$.

Proof: *(Sketch)*: The proof proceed by construting a $\Sigma_1^1(\forall\forall\exists)$ sentence Ψ (with 43 clauses!) over a certain vocabulary $\mathbf{R}$ such that a finite structure D over $\mathbf{R}$ satisfies Ψ if and only if the cardinality of the universe of D is of the form $(n^2+3n+4)/2$ for some integer n.

The construction of the above sentence Ψ uses ideas from Goldfarb's [Gol84] proof of the unsolvability of the satisfiability problem for the Gödel class. The main technical innovation in that proof was the construction of a $\forall\forall\exists$ first-order sentence ϕ that is satisfiable, but has no finite models. The Σ_1^1(Gödel) sentence Ψ that has no asymptotic probability is obtained by modifying ϕ appropriately. ∎

We can conclude that for first-order logic with equality the classifications of prefix classes according to their docility and according to the 0-1 law for the corresponding Σ_1^1 fragments are identical. This follows from the positive results for the classes Σ_1^1(Bernays-Schönfinkel) and Σ_1^1(Ackermann), and the negative results for the classes $\Sigma_1^1(\forall\exists\forall)$, $\Sigma_1^1(\forall^3)$, and $\Sigma_1^1(\forall\forall\exists)$.

Let us now consider the classification for the prefix classes without equality. Clearly, the 0-1 laws for the classes Σ_1^1(Bernays-Schönfinkel) and Σ_1^1(Ackermann) hold. On the other hand, the sentences used the demonstrate the failure of the 0-1 laws for $\Sigma_1^1(\forall\exists\forall)$, $\Sigma_1^1(\forall^3\exists)$, and $\Sigma_1^1(\forall\forall\exists)$ all used equality. To complete the classification without equality, we need to settle the status of the 0-1 law for the equality-free version of the latter three classes.

Consider first the class $\Sigma_1^1(\forall^3\exists)$. We showed earlier that it can express PARITY using equality. It turns out that without equality it can express PARITY almost surely. Consider the sentence

$$(\exists R)(\exists S)(\forall x)(\forall y)(\forall z)(\exists w)[S(x,w) \wedge (S(x,y) \wedge S(x,z) \rightarrow R(y,z)) \wedge (S(x,z) \leftrightarrow S(z,x)) \wedge \neg S(x,x) \wedge (R(x,y) \leftrightarrow (E(x,z) \leftrightarrow E(y,z)))].$$

It is shown in [KV90] that with asymptotic probability 1 this sentence is equivalent to the above $\Sigma_1^1(\forall^3\exists)$ sentence with equality that expresses PARITY, so neither sentence has an asymptotic probability. Thus, the 0-1 law fails for the class $\Sigma_1^1(\forall^3\exists)$ without equality.

A similar argument applies to the class $\Sigma_1^1(\forall\exists\forall)$. Consider the sentence

$$(\exists U)(\exists S)\forall x\exists y\forall z[(E(x,z) \leftrightarrow S(y,z)) \wedge [(E(y,z) \leftrightarrow S(x,z)) \wedge (U(x) \leftrightarrow \neg U(y)].$$

It is shown in [Ved97] that this sentence expresses PARITY almost surely, so it has no asymptotic probability. Thus, the 0-1 law fails also for the class $\Sigma_1^1(\forall\exists\forall)$ without equality. (See also [Ten94].)

So far, the classifications of prefix classes according to their docility and according to the 0-1 law for the corresponding Σ^1_1 fragments seem to agree also for prefix classes without equality. Here also the difficult case was the Gödel class. Recall that the Gödel class without equality is docile. Nevertheless, Le Bars showed that the 0-1 law for the class $\Sigma^1_1(\text{Gödel})$ without equality fails [LeB98], confirming a conjecture in [KV90]. This implies that the two classificiation do not coincide without the presence of equality.

The failure of the 0-1 law for $\Sigma^1_1(\forall\forall\exists)$ without equality is demonstrated by showing that this class can express a certain property that does not have an asymptotic probability. Recall that a set U of nodes of a directed graph $G = (V, E)$ is *independent* if there are no edges between nodes in U and *dominating* if there is an edge from each node in $V - U$ to some node of U. We say that U is a *kernel* if it is both independent and dominating. The KERNEL property says that the graph has at least one kernel. It is easy to express KERNEL in $\Sigma^1_1(\forall\forall\exists)$ without equality:

$$(\exists U)(\forall x)(\forall y)(\exists z)[((U(x) \wedge U(y)) \rightarrow \neg E(x, y)) \wedge (\neg U(x) \rightarrow (U(z) \wedge E(x, z)))].$$

The KERNEL property has asymptotic probability 1 [dlV90]. Le Bars [LeB98] defined a variant of KERNEL, using a vocabulary with 16 binary relation symbols, that is also expressible in $\Sigma^1_1(\forall\forall\exists)$ without equality. He then showed that this property does not have an asymptotic probability.

Why do the docility classification and the 0-1 classification differ on the Gödel class? As has been already established in [Gol84], the "well-behavedness" of the Gödel class is very fragile. While the Gödel class without equality is docile, the class with equality is not. Thus, the addition of one built-in relation suffices to destroy the well-behavedness of the class. In the context of the 0-1 law, we are effectively adding a built-in relation–the random graph. Apparently, adding the random graph as a built-in relation also suffices to destroy the well-behavedness of the Gödel class.

While there is some intrinsic connection between docilty and 0-1 laws (see discussion in [KV89]), the failure of the 0-1 law for the class $\Sigma^1_1(\forall\forall\exists)$ without equality shows that the two classifications need not be identical. In fact, Le Bars's result demonstrates another instance of such a divergence. Consider fragments of first-order logic defined according to the number of individual variables used. Thus, FO^k is the set of first-order sentences with at most k variables. The unsolvability of the prefix class $\forall\exists\forall$ shows that FO^3 is unsolvable. On the other hand, it is known that FO^2 is solvable [Mor75] (in fact, satisfiability of FO^2 is NEXPTIME-complete [GKV97]). The failure of the 0-1 law for the class $\Sigma^1_1(\forall\exists\forall)$ implies its failure for the class $\Sigma^1_1(\text{FO}^3)$. Le Bars, however, showed that his variant of KERNEL can be expressed in $\Sigma^1_1(\text{FO}^2)$ without equality. Thus, FO^2 is docile even with equality, but $\Sigma^1_1(\text{FO}^2)$ without equality does not have a 0-1 law.

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