Definability of Combinatorial Functions and Their Linear Recurrence Relations

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Overview

- Counting Functions and Specker Functions
- Linear Recurrence over $\mathbb{Z}$
- Order Invariance
- Linear Recurrence Relations for $\mathcal{L}^k$-Specker Function
- Linear recurrence $\Rightarrow$ c.o.i $MSOL^1$-Specker polynomial
- Main Theorem

Outline

- In this lecture we look at functions $f : \mathbb{N} \to \mathbb{N}$ which have a combinatorical interpretation.
- We then ask what can be said about their growth rate, and specially if they satisfy a linear recurrence relation.
- We introduce the concept of ordered structures, and their counting function.
- We define the Specker Polynomial, and use it to characterize functions that satisfy linear recurrence.

Counting Functions

and

Specker Functions

The counting function

- Let A be the class of all finite relational structures with relations $R_i : 1 \leq i \leq s$ of arity ρ_i (we denote it as $\bar{R}$).
- For a subclass $\mathcal{K} \subseteq A$ of structures, closed under isomorphism, we define $\mathcal{K}^n$ to be to be the structures in $\mathcal{K}$ with the universe $[n]$.
- We define the counting function for $\mathcal{K}$ as $sp_{\mathcal{K}}(n) = |\mathcal{K}^n|$.

The counting function

Examples:

(i) let $\mathcal{K}_1$ be the set of all binary relations ($s = 1; \rho_1 = 2$) then $sp_{\mathcal{K}_1}(n) = 2^{n^2}$

(ii) let $\mathcal{K}_2$ be the set of binary strings ($s = 1; \rho_1 = 1$) with exactly m "1"s, then $sp_{\mathcal{K}_2}(n) = \binom{n}{m}$.

(iii) let $\mathcal{K}_3$ be the set of full graphs ($s = 1; \rho_1 = 2$), then $sp_{\mathcal{K}_3}(n) = 1$

(iv) let $\mathcal{K}_4$ be the set of prime sized sets ($s = 0$), then $sp_{\mathcal{K}_4}(n) = 1$ if n is prime and zero otherwise.

As the examples above show, the growth rate of $sp_{\mathcal{K}}$ can be very different from one class to another.

The counting function

- For a class $\mathcal{K}$, we would like to know how the function $sp_{\mathcal{K}}(n)$ behave, specially if it satisfy a linear recurrence.

- In addition, we would like $sp_{\mathcal{K}}(n)$ to have a combinatorial interpretation.

- Therefore the current definition is too general and we need to give some constraints over our counting function.

Specker Functions

We say that $\mathcal{K}$ is definable in logic $\mathcal{L}$ if there is a sentence ϕ in $\mathcal{L}$ such that for any $\bar{R}$ structure $\mathcal{A}$, $\mathcal{A} \in \mathcal{K}$ iff $\mathcal{A} \models \phi$.

Definition (Specker function):

A function $f : \mathbb{N} \to \mathbb{N}$ is called a $\mathcal{L}^k$ Specker function if there exists a finite set of relation symbols $\bar{R}$ of arity at most k and a class $\mathcal{K}$ of $\bar{R}$-structures definable in $\mathcal{L}$ such that $f(n) = sp_{\mathcal{K}}(n)$.

Specker Functions

We would like to know, under what conditions the Specker function $sp_{\mathcal{K}}(n)$ satisfy a linear recurrence relation.

$$f(n+k) = \sum_{i=0}^{k-1} a_i f(n+i); \quad a_i \in \mathbb{Z}$$

We can use characteristic equations to show that the solution for a linear recurrence as the one above is bounded above by $f(n) \leq 2^{O(n)}$.

Specker Functions - Examples

(i) Let $f(n) = 2^n$ then f counts the number of subsets of a set of size n. This function is FOL^1 definable using a single unary relation.

(ii) We can generalize it to k^n for all $k \in \mathbb{N}$ This function has a simple linear recurrence $f(n+1) = kf(n)$

(iii) The number of linear orders n! is a FOL^2 Specker function. This function doesn't satisfy a linear recurrence because it is not $2^{O(n)}$.

(iv) The number of undirected labeled trees (due to Cayley) is n^{n-2} which is $MSOL^2$ definable.

Linear Recurrence over $\mathbb{Z}$

Linear Recurrence over $\mathbb{Z}$

- Let τ be the dictinary with unary relation.
  We can look on a structure with the universe $[n]$ as a word over binary alphabet, but then two words with the same number of "1"s will be isomorphic.

- To be able to represent general languages, we introduce the natural order $<_{nat}$ over the universe.

- Using the logical representation for words in languages we have the next theorems:

Linear Recurrence over $\mathbb{Z}$

Theorem 1 (Schützenberger)
For every regular language L, $a_L(n) = |\{w \in L : |w| = n\}|$ *satisfy a linear recurrence relation.*

Theorem 2 (Büchi)
Let $\mathcal{K}$ *be a language. Then* $\mathcal{K}$ *is regular iff it is definable in MSOL given the natural order* $<_{nat}$ *on* $[n]$

From these two theorems we get

Theorem 3
Let f *be a* $MSOL^1$ *Specker function, given the natural order* $<_{nat}$ *on* $[n]$*, then it satisfy a linear recurrence relation over* $\mathbb{Z}$

$$f(n) = \sum_{j=1}^{d} a_j f(n-j)$$

where a_j *are constant.*

Linear Recurrence over $\mathbb{Z}$

- The last theorem shows that a $MSOL^1$ Specker function f given the natural order satisfies a linear recurrence.

- We want to show a converse direction.

- In order to do so, we first extend the definition of the Specker functions, to be able to count ordered structures (the same way we added $<_{nat}$ to prove the last theorems).

Order Invariance

Order Invariance

Definition: (Order Invariant)

- A class $\mathcal{D}$ of ordered $\bar{R}$-structures is a class of $\bar{R} \cup \{<_1\}$ structures, where for every $\mathcal{A} \in \mathcal{D}$ the interpretation of the relation symbol $<_1$ is always a linear order of the universe of $\mathcal{A}$

- An $\mathcal{L}$ formula $\phi(\bar{R}, <_1)$ for ordered $\bar{R}$ structures is *truth-value order inavariant (t.v.o.i)* if for any two structures

$$\mathcal{A}_i = \left\langle [n], <_i, \bar{R} \right\rangle \ (i = 1, 2)$$

we have that

$$\mathcal{A}_1 \models \phi \iff \mathcal{A}_2 \models \phi$$

This means that if $\mathcal{A}_1 = \mathcal{A}_2$ are the same except for maybe their linear orders then they could not be told apart by ϕ.

We denote by $TV\mathcal{L}$ the set of t.v.o.i $\mathcal{L}$ formulas.

Order Invariance - Examples

a. Any binary language is a class of ordered P-structures, where P is a unary relation.
$<$ define the order of the letters in the word, and P defines what is the letter.

b. The formula $\varphi_1 =$"There are at least two "1" in the word" is $t.v.o.i$

c. The formula $\varphi_2 =$"The last letter is "1" (in the relation)" is not $t.v.o.i.$
Let $\mathcal{A}_i$ be the structures $\langle \{a, b\}, <_i, P \rangle$ where $P = \{a\}$

$$a <_1 b \text{ (the word "10")}$$

and

$$b <_2 a \text{ (the word "01")}$$

Here we have $\mathcal{A}_1 \not\models \varphi_2$ but $\mathcal{A}_2 \models \varphi_2$.

Order Invariance - Cont.

- For a class of ordered structures $\mathcal{D}$, let

  $osp_{\mathcal{D}}(n, <_1) = |\{(R_1, ..., R_s) \subseteq [n]^{\rho(1)} \times \cdots \times [n]^{\rho(s)} : \langle [n], <_1, R_1, ..., R_s \rangle \in \mathcal{D}\}|$

  A function $f : \mathbb{N} \to \mathbb{N}$ is called an $\mathcal{L}^k$-ordered Specker function if there is a class of ordered $\bar{R}$ structures $\mathcal{D}$ of arity at most k definable in $\mathcal{L}$ such that

  $$f(n) = osp_{\mathcal{D}}(n, <_1)$$

- A function $f : \mathbb{N} \to \mathbb{N}$ is called *counting order invariant (c.o.i)* $\mathcal{L}^k$-Specker function if there is a finite set of relation symbols $\bar{R}$ or arity at most k and a class of ordered $\bar{R}$-structures $\mathcal{D}$ definable in $\mathcal{L}$ such that for all linear orders $<_1$ and $<_2$ we have

  $$f(n) = osp_{\mathcal{D}}(n, <_1) = osp_{\mathcal{D}}(n, <_2)$$

Order Invariance - Examples

For a binary language define

$$\varphi_1 = \forall x \forall y (\phi_{succ}(x,y) \rightarrow (P(x) \rightarrow \neg P(y)))$$

where ϕ_{succ} means y is the succesor of x.

$$\phi_{succ}(x,y) = (x <_1 y) \wedge (\forall z[(x <_1 z \wedge z \neq y) \rightarrow y <_1 z])$$

This formula says that the word doesn't contain two consecutive "1"s.

If f is the FOL^1-ordered Specker function defined for φ_1, then an easy calculation shows that

$$f(n) = Fib(n+1)$$

Linear Recurrence Relations

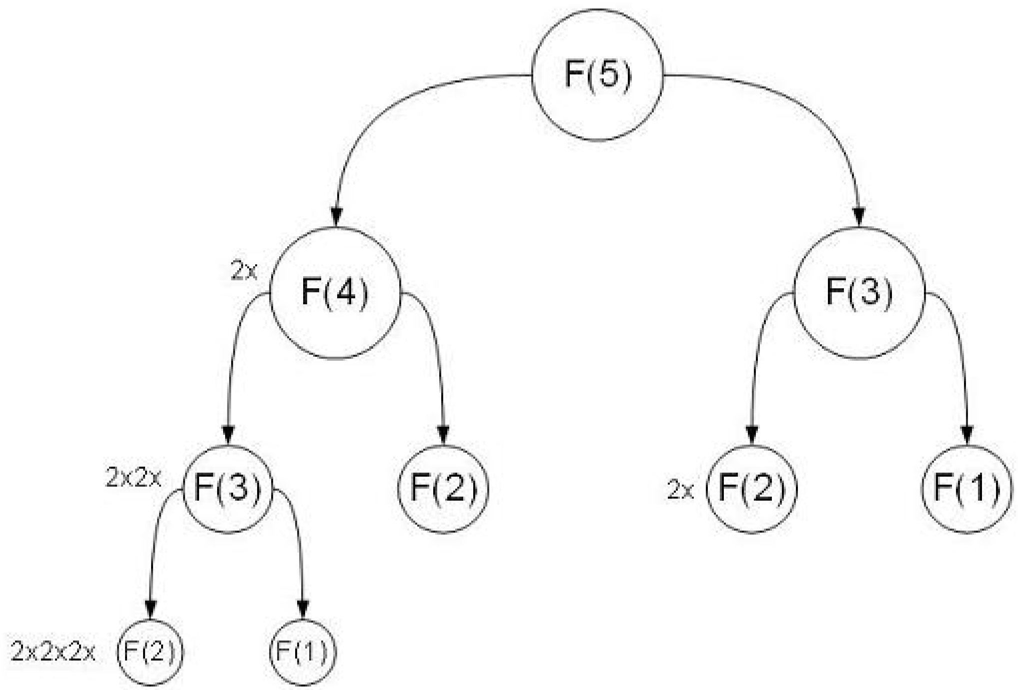
for

$\mathcal{L}^k$-Specker Function

Example - $F(n) = 2 \cdot F(n-1) + F(n-2)$

Lets first look at an example using the linear recurrence

$$F(n) = 2 \cdot F(n-1) + F(n-2)$$



Example - $F(n) = 2 \cdot F(n-1) + F(n-2)$

There are 5 routes that go from the root to one of the leaves.

$$(5,4,3,2),\ (5,4,3,1),\ (5,4,2),\ (5,3,2),\ (5,3,1)$$

The value of $F(5)$ is $2^3F(2) + F(1) + F(2) + 2F(2) + F(1)$

Each route is a sequence of choices in the recursion formula.

For example, the first route from the left, always choose $2 \cdot F(n-1)$.
The second from the left, first choose $2 \cdot F(n-1)$ and then $F(n-2)$.

Example - $F(n) = 2 \cdot F(n-1) + F(n-2)$

Each route can be thought of as a partition of the set $\{1, 2, 3, 4, 5\}$ to three sets:

(i) The set of its internal vertices. This can be subdivided into the vertices corresponding to the choice of $2F(n-1)$ and the vertices corresponding to $F(n-2)$.

(ii) The (singleton) set of the leaf.

(iii) All the other numbers that do not appear in the route.

Example - $F(n) = 2 \cdot F(n-1) + F(n-2)$

We return to the general case

$$f(n+k) = \sum_{i=0}^{k-1} a_i f(n+i); \quad a_i \in \mathbb{Z}$$

For a route P, we denote by t_i the number of times the route chose to go down in the tree using $a_i f(n+i)$, then P contribute to $f(n)$

$$a_0^{t_0} \cdot a_1^{t_1} ... a_{k-1}^{t_{k-1}} \cdot f(\beta_n) = \prod_{j=1}^{t_0} a_0 \cdot \prod_{j=1}^{t_1} a_1 ... \prod_{j=1}^{t_k} a_k \cdot f(\beta_n)$$

Where $f(\beta_n)$ is an initial condition.

In the end, we sum over all the routes. This leads us to the next definition:

$\mathcal{L}^k$-Specker polynomials

Definition:

(i) A $\mathcal{L}^k$ *Specker polynomial* $A(n, \bar{x})$ in indeterminate set $\bar{x}$ has the form

$$\sum_{R_1:\Phi_1(R_1)} \cdots \sum_{R_t:\Phi_t(R_1,\ldots,R_t)} \left(\prod_{v_1,\ldots,v_k:\Psi_1(\bar{R},\bar{v})} x_{m_1} \cdots \prod_{v_1,\ldots,v_k:\Psi_l(\bar{R},\bar{v})} x_{m_l} \right)$$

where $\bar{v}$ stands for $(v_1, ..., v_k)$, $\bar{R}$ stands for $(R_1, ..., R_t)$ and the R_i's are relation variables of arity ρ_i at most k.

Each R_i range over relations of arity ρ_i over $[n]$ and the v_i range over elements of $[n]$ satisfying the iteration formulas $\Phi_i, \Psi_i \in \mathcal{L}$

(ii) Order invariant $\mathcal{L}^k$-Specker polynomial are defined analogously to Specker functions.

$\mathcal{L}^k$-Specker polynomials - Examples

- If f is a Specker function that counts $\bar{R}$-structure which satisfy $\varphi(\bar{R})$, then we can write $f(n) = \sum_{\bar{R}:\varphi(\bar{R})} 1$ so the Specker polynomial is a generalization of the Specker function.

- Let P be a unary relation then the Specker Polynomial

$$\sum_{P:true(P)} \prod_{v:P(v)} x = \sum_{P:true(P)} x^{|P|} = \sum_{m=0}^{n} \binom{n}{m} x^m$$

is the generating function for the binom function $f_n(m) = \binom{n}{m}$.

$\mathcal{L}^k$-Specker polynomials

First we show that Specker Polynomials behave nicely under variable substitution with polynomials.

Lemma 4
Let $A(n, \bar{z})$ ba a $c.o.i$ $MSOL^1$-Specker polynomial with indeterminates $\bar{z} = (z_1, ..., z_s)$ where $h_i(\bar{w}) \in \mathbb{Z}$.

Let $A(n, (h_1(\bar{w}), ..., h_s(\bar{w})))$ denote the variable substitution in $A(n, \bar{z})$ where for $i \in [s]$, z_i is substituted to $h_i(\bar{w})$.

Then $A(n, \bar{h})$ is an integer evaluation of a counting order invariant $MSOL^1$-Specker polynomial.

Proof of Lemma 4

We show the lemma is true for substituting z_1 by $h_1(\bar{w})$, and the lemma will follow by induction. Write

$$h_1(\bar{w}) = \sum_{j=1}^{d} c_j w_1^{\alpha_{j,1}} \cdots w_t^{\alpha_{j,t}}$$

The indeterminate z_1 shows in the Specker polynomial as $\prod_{v_1 : \Psi_1(\bar{R}, v_1)} z_1$.

After substituting z_1 by $h_1(\bar{w})$, we will get

$$\left(\sum_{j=1}^{d} c_j w_1^{\alpha_{j,1}} \cdots w_t^{\alpha_{j,t}}\right) \left(\sum_{j=1}^{d} c_j w_1^{\alpha_{j,1}} \cdots w_t^{\alpha_{j,t}}\right) \cdots \left(\sum_{j=1}^{d} c_j w_1^{\alpha_{j,1}} \cdots w_t^{\alpha_{j,t}}\right)$$

each monom in the result of the above product, is a multiplication of monoms from each of the copies of h_1.

Proof of Lemma 4 - Cont.

In other words, each monom in the result can be written as

$$\prod_{v_1 \in U_1} c_1 w_1^{\alpha_{1,1}} \cdots w_t^{\alpha_{1,t}} \cdot \prod_{v_1 \in U_2} c_2 w_1^{\alpha_{2,1}} \cdots w_t^{\alpha_{2,t}} \cdots \prod_{v_1 \in U_d} c_d w_1^{\alpha_{d,1}} \cdots w_t^{\alpha_{d,t}}$$

where $(U_1, U_2, ..., U_d)$ is a partition of the elements of $[n]$ which satisfy $\Psi_1(\bar{R}, v_1)$ - we denote this formula by $\phi_{Part(\Psi_1)}(\bar{U})$.

Now we only need to sum over all partitions, so

$$A(n, (h_1(\bar{w}), z_2, ..., z_s)) =$$

$$\sum_{R_1 : \Phi_1(R_1)} \cdots \sum_{R_t : \Phi_t(\bar{R})} \sum_{\bar{U} : \phi_{Part(\Psi_1)}(\bar{U})}$$

$$\left(\prod_{v_1 : \Psi_2(\bar{R}, v_1)} z_2 \cdots \prod_{v_1 : \Psi_s(\bar{R}, v_1)} z_s \cdot \prod_{v_1 \in U_1} c_1 w_1^{\alpha_{1,1}} \cdots w_t^{\alpha_{1,t}} \cdots \prod_{v_1 \in U_d} c_d w_1^{\alpha_{d,1}} \cdots w_t^{\alpha_{d,t}} \right)$$

where $\bar{U} = (U_1, U_2, ..., U_d)$. Notice that $\phi_{Part(\Psi_1)}(\bar{U})$ is in $MSOL^1$.

Proof of Lemma 4 - Cont.

To complete the proof we see that

$$\prod_{v_1:\theta} c_j w_1^{\alpha_{j,1}} \cdots w_t^{\alpha_{j,t}} = \prod_{v_1:\theta} c_j \left(\overbrace{\prod_{v_1:\theta} w_1 \cdots \prod_{v_1:\theta} w_1}^{a_{j,1}\ times} \right) \cdots \left(\overbrace{\prod_{v_1:\theta} w_t \cdots \prod_{v_1:\theta} w_t}^{a_{j,t}\ times} \right)$$

We can now substitute all the c_i with new indeterminates, and so $A(n, (h_1(\bar{w}), z_2, ..., z_s))$ is an evaluation of an counting order invariant $MSOL^1$-Specker polynomial.

Linear recurrence ⇒ c.o.i $MSOL^1$-Specker polynomial

Linear recurrence $\Rightarrow$ c.o.i $MSOL^1$-Specker polynomial

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The next theorem shows the connection between polynomials with linear recurrence and and the Specker polynomials

Theorem 5
Let $A_n(\bar{x})$ be a sequence of polynomials with finite indeterminate set $\bar{x} = (x_1, ..., x_s)$ which satisfies a linear recurrence over $\mathbb{Z}[\bar{x}]$.

Then, there exists a counting order invariant $MSOL^1$-Specker polynomial $A'(n, \bar{x}, \bar{y})$ such that $A_n(\bar{x}) = A'(n, \bar{x}, \bar{a})$ where $\bar{a} = (a_1, ..., a_l) \in \mathbb{Z}^l$.

Linear recurrence $\Rightarrow$ c.o.i $MSOL^1$-Specker polynomial

Proof:

$A_n(\bar{x})$ has linear recurrence so we write

$$A_n(\bar{x}) = \sum_{i=1}^{r} f_i(\bar{x}) A_{n-i}(\bar{x}) \qquad f_i(\bar{x}) \in \mathbb{Z}[\bar{x}]$$

$$A_1(\bar{x}), A_2(\bar{x}), ..., A_r(\bar{x}) \in \mathbb{Z}[\bar{x}]$$

In order to evaluate $A_n(\bar{x})$ we need first to evaluate $A_{n-i_1}(\bar{x})$ for all $i_1 \in 1, ..., r$.

In the same manner, in order to evaluate $A_{n-i_1}(\bar{x})$ we need first to evaluate $A_{n-i_1-i_2}(\bar{x})$ for all $i_2 \in 1, ..., r$.

We can continue this until we reach an initial condition.

Linear recurrence ⇒ c.o.i $MSOL^1$-Specker polynomial

We define a path in the recurrence tree to be

$$(A_n(\bar{x}), A_{n-i_1}(\bar{x}), A_{n-i_1-i_2}(\bar{x}), ..., A_{n-i_1-i_2-...-i_l}\bar{x})$$

where $i_k \in [r]$ and $A_{n-i_1-i_2-...-i_l}(\bar{x})$ is an initial condition.

Finally, to evaluate $A_n(\bar{x})$ we need to sum up over all the recurrence pathes.

Linear recurrence $\Rightarrow$ c.o.i $MSOL^1$-Specker polynomial

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A recurrence path is defined by the numbers $T = (i_1, i_2, ..., i_l)$.

It contributes

$$f_1(\bar{x})^{\#(1,T)} \cdots f_r(\bar{x})^{\#(r,T)} A_{n-\sum i_k}(\bar{x})$$

where $\#(j,T)$ counts the number of times j appears in T.

For each path we partition $[n]$ to three subsets

- $\bar{U} = (U_1, U_2, ..., U_r)$ - where U_j is all the k such that A_k appears in the path where the next polynomial in the path is A_{k-j} (and in particular we get that $k \notin [r]$ - it's not one of the initial conditions)
- $\bar{I} = (I_1, I_2, ..., I_r)$ - where $I_j = \{j\}$ if A_j is the initial condition in the path, and $I_j = \emptyset$ otherwise.
- $S = [n] \setminus (\bigcup U_j \cup \bigcup I_j)$ - the rest of $[n]$.

Linear recurrence ⇒ c.o.i $MSOL^1$-Specker polynomial

We now write $A_n(\bar{x})$ as

$$A_n(\bar{x}) = \sum_{\bar{U},\bar{I},S,\phi_{rec}(\bar{U},\bar{I},S)} \prod_{v:v\in U_1} f_1(\bar{x}) \cdots \prod_{v:v\in U_r} f_r(\bar{x}) \cdot \prod_{v:v\in I_1} A(1,\bar{x}) \cdots \prod_{v:v\in I_r} A(r,\bar{x})$$

Where $\phi_{rec}(\bar{U},\bar{I},S)$ says

(i) $\left(\bar{U},\bar{I},S\right)$ form a partition for $[n]$.

(ii) $n \in \bigcup U_j$ - the recurrence path start with n.

(iii) $|\bigcup I_j| = 1$ - there is only one initial condition in the path.

(iv) if $v \in [n] \setminus [r]$ then $v \notin \bigcup I_j$, and if $v \in [r]$ then $v \notin \bigcup U_j$ - the path must go through vertices in $\bar{U}$ and reach an end when $v \in [r]$.

(v) for every $v \in U_j$, the vertices $\{v-1, v-2, ..., v-(j-1)\}$ are in S, and $v - j \in (\bigcup U_j \cup \bigcup I_j)$

Linear recurrence $\Rightarrow$ c.o.i $MSOL^1$-Specker polynomial

Next, we define

$$B(n,\bar{x}) = \sum_{\bar{U},\bar{I},S,\phi_{rec}(\bar{U},\bar{I},S)} \prod_{v:v\in U_1} z_1 \cdots \prod_{v:v\in U_r} z_r \cdot \prod_{v:v\in I_1} z_{r+1} \cdots \prod_{v:v\in I_r} z_{2r}$$

ϕ_{rec} is $MSOL^1$ definable, so $B(n,\bar{z})$ is a counting order invariant $MSOL^1$-Specker polynomial.

By the previous lemma we have that

$$A_n(\bar{x}) = B(n,(f_1(\bar{x}),...,f_r(\bar{x}),A(1,\bar{x}),...,A(r,\bar{x})))$$

is an evaluation in $\mathbb{Z}$ of a c.o.i $MSOL^1$-Specker polynomial.

Main Theorem

Main Theorem

At last, we wish to characterize the functions $f : \mathbb{N} \to \mathbb{N}$ which satisfy a linear reccurence.

Theorem 6
Let f be a function over $\mathbb{N}$. Then f satisfies a linear recurrence relation over $\mathbb{Z}$ iff

$$f = f_1 - f_2$$

where f_1, f_2 are two counting order invariant $MSOL^1$-Specker functions.

$f = f_1 - f_2$; f_1, f_2 c.o.i $\Rightarrow f$ satisfy linear recurrence

Proof:
First assume that $f = f_1 - f_2$ where f_1, f_2 are c.o.i $MSOL^1$-Specker functions.

From theorem 3 we have that f_1, f_2 satisfies a linear recurrecne relation over $\mathbb{Z}$.

It can be shown that the difference of two linear recurrence series also has linear recurrence relation, thus we get that the function f satisfies a linear recurrence over $\mathbb{Z}$.

f satisfy linear recurrence $\Rightarrow f = f_1 - f_2$; f_1, f_2 c.o.i

Assume now that f has a linear recurrence over $\mathbb{Z}$.

From the previous theorem, f is an evaluation of a c.o.i $MSOL^1$-Specker polynomial

$$f(n) = A(n, \bar{a});\ \bar{a} = (a_1, a_2, ..., a_l) \in \mathbb{Z}^l$$

The main idea behind the proof is that for a set Y, and a non negative integer $a \geq 0$, $a^{|Y|}$ counts the number of partition of Y to a disjoint subsets.

We shall prove the theorem for one indeterminate, and the general case is similar.

f satisfy linear recurrence $\Rightarrow f = f_1 - f_2$; f_1, f_2 c.o.i

Let

$$A(n,y) = \sum_{R:\Phi(R)} \prod_{v:\Psi(R,v)} y$$

We define the set Y to be all the v such that $\Psi(R,v)$.

This can be defined in $MSOL^1$ by

$$\Psi'(Y,R) = \forall v (v \in Y \iff \Psi(R,v))$$

For a non negative $a \geq 0$ denote $\bar{Z} = (Z_1, Z_2, ..., Z_a)$ then the formula $\phi_{part}(Y, \bar{Z})$ which says that Y is the disjoint union of Z_i is $MSOL^1$ definable.

f satisfy linear recurrence $\Rightarrow f = f_1 - f_2$; f_1, f_2 c.o.i

As stated before we have for a fixed Y the equation

$$a^{|Y|} = |\{\bar{Z} \mid \phi_{part}(Y, \bar{Z})\}| = \sum_{\bar{Z}:\phi_{part}(Y,\bar{Z})} 1$$

We now sum it all up

$$\begin{aligned}
A(n,a) &= \sum_{R:\Phi(R)} \prod_{v:\Psi(R,v)} a &&= \sum_{R,Y:\Phi(R)\wedge\Psi'(Y,R)} \prod_{v:v\in Y} a \\
&= \sum_{R,Y:\Phi(R)\wedge\Psi'(Y,R)} a^{|Y|} &&= \sum_{R,Y,\bar{Z}:\Phi(R)\wedge\Psi'(Y,R)\wedge\phi_{part}(Y,\bar{Z})} 1 \\
&= \sum_{R,Y,\bar{Z}:\beta_a(R,Y,\bar{Z})} 1 &&= |\{R, Y, \bar{Z} \mid \beta_a(R, Y, \bar{Z})\}|
\end{aligned}$$

where $\beta_a(R, Y, \bar{Z}) = \Phi(R) \wedge \Psi'(Y, R) \wedge \phi_{part}(Y, \bar{Z})$

f satisfy linear recurrence $\Rightarrow f = f_1 - f_2$; f_1, f_2 c.o.i

This shows that if f is the evaluation at a non-negative $a \geq 0$, then it is a c.o.i $MSOL^1$ Specker Polynomial.

Because the constant function 0 is also a c.o.i $MSOL^1$ Specker function, then f is the difference of two c.o.i $MSOL^1$ Specker function.

If $a < 0$ in a similar way we get

$$\begin{aligned} A(n,a) &= \sum_{R,Y,\bar{Z}:\beta_{|a|}(R,Y,\bar{Z})} (-1)^{|Y|} \\ &= \sum_{R,Y,\bar{Z}:\beta_{|a|}(R,Y,\bar{Z})\wedge\phi_{even}(Y)} 1 - \sum_{R,Y,\bar{Z}:\beta_{|a|}(R,Y,\bar{Z})\wedge\neg\phi_{even}(Y)} 1 \\ &= |\{R,Y,\bar{Z} \mid \beta_{|a|}(R,Y,\bar{Z}) \wedge \phi_{even}(Y)\}| \\ &\quad -|\{R,Y,\bar{Z} \mid \beta_{|a|}(R,Y,\bar{Z}) \wedge \neg\phi_{even}(Y)\}| \end{aligned}$$

Using the order $<$ we have on $[n]$ we can write ϕ_{even} in $MSOL^1$, and so we have that for all $a \in \mathbb{Z}$, the evaluation at a is the difference of two c.o.i $MSOL^1$ Specker function.

Main Theorem - Examples

(i) The Fibonacci sequence satisfy $F(n+2) = F(n+1) + F(n)$, hence it is the difference of two c.o.i $MSOL^1$ Specker function.

(ii) This theorem generalize theorem 3 - the function $g \equiv 0$ is c.o.i $MSOL^1$ Specker function, so every c.o.i $MSOL^1$ Specker function f satisfy a linear recurrence relation, since $f = f - 0 = f - g$.