0-1 Law for Existential Monadic Second Order Logic: Counter Examples

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Outline, I

- Introduction

- 0-1 law fails for $MESO$

- 0-1 law fails for $MESO$ on undirected graphs

- Additional remarks

- References

Outline, II

First we give a reminder of the 0-1 law. We define ESO and $MESO$ and give a simple counter example to prove that the 0-1 law fails for ESO.
Next we prove that the 0-1 law still fails for $MESO$. Afterwards we see how to modify the proof so the failure of $MESO$ on undirected graphs is established.
Finally we present restrictions that can be considered on ESO (and $MESO$).

Introduction

- 0-1 law: A reminder
- Definitions: ESO and $MESO$
- 0-1 law fails for ESO

0-1 law: A reminder

Let $\mathcal{R}$ be a vocabulary and $\mathcal{P}$ be a property on the collection of all finite structures over $\mathcal{R}$.

- $\mu_n(\mathcal{P})$ denotes the fraction of finite models with domain $n = \{0, ..., n-1\}$.
- $\mu(\mathcal{P}) = \lim_{n\to\infty}\mu_n(\mathcal{P})$ is called the asymptotic probability of $\mathcal{P}$.
- We say that the 0-1 law holds for a logic if the asymptotic probability of any property which is expressible in this logic is either 0 or 1.

Definitions: ESO and $MESO$

- An existential second-order sentence over a vocabulary $\mathcal{R}$ is an expression ψ of the form $\exists \mathcal{S} \varphi(\mathcal{R} \cup \mathcal{S})$, where $\mathcal{S}$ is a set of relation variables and $\varphi(R \cup S)$ is a first-order sentence over $\mathcal{R} \cup \mathcal{S}$.

- Existential second-order logic, denoted by ESO, is the set of such expressions.

- The sentence ψ is said to be monadic when $\mathcal{S}$ is a set of unary relation variables.

- Similarly, $MESO$, monadic existential second-order logic, denotes the set of such expressions.

0-1 law fails for ESO

- We need to find an ESO sentence which has no limit probability (or which is not equal to 0 or 1).
- $PARITY$ is definable in ESO and has no limit probability.
- The following sentence is in ESO and it expresses $PARITY$:

  $$\exists S \forall x \exists y \forall z (S(x,y) \wedge \neg S(x,x) \wedge S(x,z) \rightarrow y = z \wedge S(x,z) \leftrightarrow S(z,x))$$

  This sentence says that there is a permutation S in which every element has order 2.
- S is not monadic, so this is not a counter example for $MESO$. However, $PARITY$ is definable in $MESO$ as we will see in the next section.
- We will meet this sentence again in the Additional remarks section.

0-1 law fails for $MESO$

- Idea of the proof
- Definitions
- Lemmas 1-3
- Main Lemma
- Theorem 1
- Remarks

Idea of the proof

- We will prove that the 0-1 law fails for $MESO$ by defining $PARITY$.
- It will be easy to define $PARITY$ once we have a linear order of the universe. This is the Main Lemma.
- The proof of the Main Lemma consists on extending a linear order from a small part of the structure to the whole structure by using several times the lexicographic order.
- This is done by using Lemmas 1-3. We use Lemma 1 to say that the existence of the small part is almost always true (it has limit probability 1). Lemmas 2 and 3 are used to say that it is almost always true that we can extend the linear order.

Definitions

Let A and B be subsets of a structure $(C; R, ...)$, where R is binary.

(i) For $b \in B$, we say that b R-codes $\{a \in A :< a, b >\in R\}$ with respect to A.

(ii) We say that B codes distinct subsets of A if no two elements of B code the same subset of A.

(iii) We say that B codes the power set of A if B codes distinct subsets of$A$ and moreover every subset of A is coded by an element of B.

Lemma 1, I

Lemma 1. Let R be an arbitary binary relation on $\{0, 1, ..., k-1\}$, and let n be an integer greater than $k^2 \cdot 4^k$. Let p be the probability that some substructure of a random model of the form $(\{0, ..., n-1\}; R')$ contains an isomorphic copy of $(\{0, 1, ..., k-1\}; R)$. Then p approaches 1 as k approaches infinity.

Lemma 1, II

Proof:

- We build the requested isomorphic embedding.
- Let $a = k \cdot 4^k$, $A = \{0, ..., a \cdot k\} \subseteq \{0, ..., n-1\}$.
- Partition A into k pieces each of size a.
- At each step $1 \leq i \leq k$ attemp to extend the embedding by mapping i to some element of the i^{th} piece of the partition.
- Assume we try to add e at step i. Let $\{e_1, ..., e_{i-1}\}$ be the elements that were already chosen. The probability that e lies in the appropriate relation with itself is $\frac{1}{2}$, and for each e_j, $1 \leq j \leq i-1$, it is $\frac{1}{4}$. Thus the probability of failure at step i is $(1 - \frac{1}{2} \cdot (\frac{1}{4})^{i-1})^a \leq (1 - (\frac{1}{4})^k)^a$.
- Hence the probability of failure is bounded above by $k \cdot (1 - (\frac{1}{4})^k)^a = k \cdot (1 - (\frac{1}{4})^k)^{k \cdot 4^k}$, which approaches 0 as k approaches infinity.

Lemma 2, I

Lemma 2. If $S \subseteq T \subseteq A$, where $(A; R, ...)$ is a finite structure, and if $|T| \geq |S| \cdot 2^{|S|}$ then with limit probability 1, some subset T' of T codes the power set of S.

Lemma 2, II

Proof:

- We show that with limit probability 1, every subset S_i, $1 \leq i \leq 2^{|S|}$, is coded by (at least) one element, e_i, of T. Then $T' = \{e_i : 1 \leq i \leq 2^{|S|}\}$.

- Let S' be a subset of S.

- For each $t \in T$ the probability it codes S' is $\frac{1}{2^{|S|}}$. Thus the probability that S' is not coded by an element of T is $(1 - \frac{1}{2^{|S|}})^{|T|} \leq (1 - \frac{1}{2^{|S|}})^{|S| \cdot 2^{|S|}}$.

- Hence, using union-bound, the probability that there is a subset of S which is not coded by any element of T is bounded by $2^{|S|} \cdot (1 - \frac{1}{2^{|S|}})^{|S| \cdot 2^{|S|}}$.

- The second factor is asymptotic with $\frac{1}{e^{|S|}}$, so the limit is 0.

Lemma 3, I

Lemma 3. Suppose that S and T are subsets of a structure $(A; R, S, T, ...)$ in which T codes distinct subsets of S and such that there is a first-order definable total order $<$ on S. Then there is a first-order definable total order on T.

Proof: We define a total order $<<$ on T as follows: $x << y$ iff $x \neq y$ and for a equal to the $<$-least member of the symmetric difference of the sets coded by x and y, a is not in x.

Lemma 3, II

- By its defenition, $<<$ is not reflexive.
- $<<$ is total since for distinct $x_1, x_2 \in T$, $x_1 << x_2$ or $x_2 << x_1$ (the symmetric difference is not empty because T codes distinct subsets of S). Also the $<$-least element in the symmetric difference does not belong to x_1 and x_2, so $<<$ is not symmetric.
- $<<$ is transitive. For $x, y, z \in T$ s.t. $x << y$ and $y << z$ we need to prove that $x << z$.
  Let X, Y and Z be the subsets coded by x, y and z respectively. For $U \subseteq T$ let m_U be the $<$-least element in U.
  Assume by contradiction that $z << x$, so $m_{X \oplus Z} \in X$.
  If $m_{X \oplus Z} \in Y$:
  $m_{X \oplus Z} \in Y \oplus Z \Rightarrow m_{Y \oplus Z} < m_{X \oplus Z} \Rightarrow m_{Y \oplus Z} \in X \Rightarrow m_{Y \oplus Z} \in X \oplus Y \Rightarrow m_{X \oplus Y} < m_{Y \oplus Z} \Rightarrow m_{X \oplus Y} \in Z \Rightarrow m_{X \oplus Y} \in X \oplus Z \Rightarrow m_{X \oplus Z} < m_{X \oplus Y} \Rightarrow m_{X \oplus Y} < m_{X \oplus Y}$.
  Similarly, we can get a contradiction if $m_{X \oplus Z}$ is not in Y.

Main Lemma, I

Main Lemma. There is a first-order formula $\phi(x, y)$ in a vocabulary which includes a sequence of unary relation symbols $\overline{P}$ and binary relation symbols R, R_0, R_1 and R_2 such that the following sentence has limit probability 1:

$(\exists \overline{P})$ [1] "$\phi(x, y)$ defines a linear order of the universe"

[1] $\forall xy[(\phi(x,y) \wedge \neg\phi(y,x)) \vee (\phi(y,x) \wedge \neg\phi(x,y))] \bigwedge \forall xyz[(\phi(x,y) \wedge \phi(y,z)) \rightarrow \phi(x,z)]$

Main Lemma, II

Proof:

- Fix a structure $(A; R, R_0, R_1, R_2)$, and pick k s.t. $|A| \geq 2^k \cdot 2^{2^k} = 2^k \cdot 4^k$ and $|A| < 2^{k+1} \cdot 2^{2^{k+1}}$.

- $2^k \cdot 4^k$ exceeds $k^2 \cdot 4^k$ for sufficiently large k. Thus, by Lemma 1, we choose (with limit probability 1) $P_0 \subseteq A$ of power k s.t. the restriction of R to P_0 is a total order.

- By Lemma 2 we choose (with limit probability 1) $P_1 \subseteq A$ which R_0-codes the power set of P_0 ($|A| \geq k \cdot 2^k = |P_0| \cdot 2^{|P_0|}$), and then $P_2 \subseteq A$ which R_1-codes the power set of P_1 ($|A| \geq 2^k \cdot 2^{2^k} = |P_1| \cdot 2^{|P_1|}$).

- Define $P_3 = A$. Let $\{d_1, d_2\}$ be a fixed pair of distinct elements of P_3.

- d_1 and d_2 R_2-code the same subset of P_2 with probability $2^{-|P_2|} = 2^{-2^{2^k}}$.

- The number of pairs $\{d_1, d_2\}$ is less than $n^2 < 2^{2 \cdot (k+1)} \cdot 2^{2^{k+2}}$.

Main Lemma, III

- Thus the probability of having a pair $\{d_1, d_2\}$ which R_2-codes the same subset of P_2 is less than $2^{-2^{2^k}} \cdot 2^{2\cdot(k+1)} \cdot 2^{2^{k+2}} = 2^{-(4^k - 4\cdot 2^k - 2\cdot(k+1))}$.

- This term goes to 0 as n goes to infinity, so with probability 1 P_3 R_2-codes distinct subsets of P_2.

- P_{i+1} R_i-codes distinct subsets of P_i for $i = 0, 1, 2$. Thus by successive applications of Lemma 2, there is a formula in the vocabulary $\{R, R_0, R_1, R_2, P_0, P_1, P_2\}$ which defines a total order of the universe, and this is the desired formula $\phi(x, y)$.

Main Lemma, IV

We show how to construct the formula ϕ according to the proof of the Main Lemma:

- $\mathbf{LO_0(a_0, b_0) \triangleq R(a_0, b_0)}$
  The linear order on P_0 (according to Lemma 1).

- For $1 \leq i \leq 3$:
  - $\mathbf{CS_i(x, a) \triangleq R_{i-1}(a, x) \wedge P_{i-1}(a)}$
    a belongs to the set that x R_{i-1}-codes with respect to P_{i-1}.
  - $\mathbf{SYM_i(x, y, a) \triangleq (CS_i(x, a) \wedge \neg CS_i(y, a)) \vee (CS_i(y, a) \wedge \neg CS_i(x, a))}$
    a belongs to the symmetric difference between the sets that x and y R_{i-1}-code with respect to P_{i-1}.
  - $\mathbf{LO_i(a_i, b_i) \triangleq (a_i \neq b_i) \wedge \forall a_{i-1}[[SYM_i(a_i, b_i, a_{i-1}) \wedge \neg \exists b_{i-1}(SYM_i(a_i, b_i, b_{i-1}) \wedge LO_{i-1}(b_{i-1}, a_{i-1}))] \rightarrow \neg CS_i(a_i, a_{i-1})]}$
    The linear order on P_i (constructed according to Lemma 3).

- $\mathbf{\phi(x, y) \triangleq LO_3(x, y)}$

Main Lemma, V

The number of first-order variables:

- LO_0 uses 2 first-order variables, and for $1 \leq i \leq 3$, LO_i uses LO_{i-1} variables and adds another 2.
- $\phi(x, y)$ uses 8 first-order variables.
- The sentence in the Main Lemma uses 9 first-order variables (it uses $\phi(x, y)$ variables and adds another 1).

Theorem 1

Theorem 1. There is a sentence of $MESO$ which has no limit probability.

Proof: We use the following sentence, where ϕ is as in the Main Lemma. It says of a finite structure that its universe has an odd number of elements.

$(\exists \overline{P})(\exists Q)[$"$\phi(x,y)$ defines a linear order of the universe s.t. Q contains every other element, including the first and last"$]$

Remarks:

- The formula: $\forall x[(\neg\exists y(\phi(x,y) \vee \phi(y,x))) \rightarrow Q(x)] \wedge \forall xy(\phi(x,y) \rightarrow ((Q(x) \vee Q(y)) \wedge \neg(Q(x) \wedge Q(y)))]$

- We use 9 first-order variables.

Remarks, I

- We can get a sentence of limit probability $\frac{1}{2}$ by modifying the previous sentence to say that Q contains every other element of the restriction of this linear order to an arbitry set S (a unary relation symbol of the vocabulary), including the first and last elements of S.

- The number of options to choose a subset S from the universe that has an even power is equal to this number when the subset S we choose has an odd power. The sentence above says that S has an odd power, so its limit probability is indeed $\frac{1}{2}$.

- This idea can be extended to prove the following theorem:

  Theorem 2. For every relational number r in the interval $[0, 1]$ there is a sentence of $MESO$ which has limit probability r.

Remarks, II

- A universal monadic second-order sentence over a vocabulary $\mathcal{R}$ is an expression ψ of the form $\forall \mathcal{S} \varphi(\mathcal{R} \cup \mathcal{S})$, where $\mathcal{S}$ is a set of unary relation variables and $\varphi(\mathcal{R} \cup \mathcal{S})$ is a first-order sentence over $\mathcal{R} \cup \mathcal{S}$.

- We can get an example of a sentence in this set that has no limit probability by using the sentence in the proof of Theorem 1:

  $(\forall \overline{P})(\forall Q)\neg$["$\phi(x, y)$ defines a linear order of the universe s.t. Q contains every other element, including the first and last"]

- Hence the 0-1 law fails for the set of such expressions.

0-1 law fails for $MESO$ on undirected graphs

- How to reduce the number of the binary relations

- A counter example on undirected graphs

How to reduce the number of the binary relations

- First of all, we want to reduce the number binary relations since undirected graphs have one binary relation. In the previous section we used four binary relations: R, R_0, R_1 and R_2.
- We considered distinct binary relations because of the hypothesis $S \subseteq T$ in Lemma 2. We proved for $i = 1, 2$ that P_i R_{i-1}-codes the power set of P_{i-1} and P_3 R_2-codes distinct subsets of P_2. We had $P_0 \subset P_1 \subset P_2 \subset P_3 = A$.
- We need P_i $0 \le i \le 3$ to be disjoint subsets.
- We change Lemma 2 to Lemma 4 below (with the same proof):
  Lemma 4. If S and T are disjoint subsets of A where $(A; R, ...)$ is a finite structure, and if $|T| \geq |S| \cdot 2^{|S|}$ then with limit probability 1, some subset T' of T codes the power set of S.
- The proof of Theorem 1 is similar, we just use Lemma 4 instead of Lemma 2 and define $P_3 = A \setminus (P_0 \cup P_1 \cup P_2)$ (P_3 codes distinct subsets of P_2 using the same proof), so the vocabulary $\mathcal{R} = \{R\}$.

A counter example on undirected graphs, I

- Now our $\mathcal{R}$-structure is an undirected graph $G_n =< V_n, E_n >$.
- The only place in the proof that has to be changed is where we use Lemma 1 in the proof of the Main Lemma.
- E is symmetric, we can't choose (with limit probability 1) $P_0 \subseteq V$ of power k s.t. the restriction of E to P_0 is a total order.
- However, we can use Lemma 1 to choose (with limit probability 1) $P_0 \subseteq V$ of power k s.t. the restriction of E to P_0 is a graph on which we can easily define a total order.
- We prove the following Lemma 5:
  Lemma 5. There is a sequence H_m, $m \geq 2$ of undirected graphs of cardinality m and a $MESO$ sentence $\exists V \exists W \psi(V, W)$, where V and W are unary relation variables, which defines a linear order of the vertices over these graphs.

  Thus the graph we need is H_k.

A counter example on undirected graphs, II

Proof:

- First we define the sentence $\exists V \exists W \psi(V, W)$ over undirected graphs.
- Let v and v' be distinct elements of V and W_v and $W_{v'}$ be the subsets of W E-coded respctively by v and v'.
- $\forall w((W(w) \wedge E(v, w)) \rightarrow E(v', w))$ expresses that $W_v \subseteq W_{v'}$ and $\exists w(W(w) \wedge E(v', w) \wedge \neg E(v, w))$ expresses that this inclusion is strict.
- Let $\varphi_{\prec}(W, v, v')$ denote the conjuction of this two formulas. By inverting the role of V and W, we define similarly $\varphi_{\prec}(V, w, w')$, for distinct elements w, w' of W.
- Let $\psi(V, W)$ denote a first-order formula which expresses that (V, W) form a partition of the set of vertices and $\varphi_{\prec}(W, v, v')$ (resp. $\varphi_{\prec}(V, w, w')$) is a linear order over V (resp. W).
- Provided that an undirected graph satisfies $\exists V \exists W \psi(V, W)$, we obtain a linear order over the whole domain by choosing, for example, $v \prec w$, for $v \in V$ and $w \in W$.

A counter example on undirected graphs, III

- Now we exhibit an undirected bipartite graph H_m of cardinality m, for any $m \geq 2$.

- Suppose $m = 2l$, the set of vertices of H_m is divided into two sets $V = \{v_1, ..., v_l\}$ and $W = \{w_1, ..., w_l\}$ and there is no edge between vertices of V and between vertices of W.

- We add edges between V and W as follows: v_i is adjacent to $w_1, ..., w_{l-i+1}$ for $i = 1...l$. Observe that this construction is symmetric, we obtain the same graph if we invert V with W. (for $1 \leq j, i \leq l$ there is an edge between w_j and v_i iff $1 \leq j \leq l - i + 1$, so we get $1 \leq i \leq l - j + 1$).

- Suppose $m = 2l + 1$, we build H_m from H_{2l} defined above by adding an isolated vertex v_{l+1} in V.

- For $v_i, v_j \in V$ s.t. $i < j$, $\{w_1, ..., w_{l-j+1}\} = W_j \subset W_i = \{w_1, ..., w_{l-i+1}\}$, and similarly for two vertices in W.

- $\exists V \exists W \psi(V, W)$ holds for H_m.

A counter example on undirected graphs, IV

Remark. The changes in ϕ:

- Instead of R, R_0, R_1, R_2 we write E.
- LO_0 is different:
  $LO_0(x,y) = (V(x) \wedge W(y)) \vee (V(x) \wedge V(y) \wedge \varphi_{\prec}(V,x,y)) \vee (W(x) \wedge W(y) \wedge \varphi_{\prec}(W,x,y))$
- ϕ is different (it is not LO_3):
  $\phi(x,y) = (P_0(x) \wedge \neg P_0(y)) \vee (P_1(x) \wedge \neg P_0(y) \wedge \neg P_1(y)) \vee (P_2(x) \wedge P_3(y)) \vee (P_0(x) \wedge P_0(y) \wedge LO_0(x,y)) \vee (P_1(x) \wedge P_1(y) \wedge LO_1(x,y)) \vee (P_2(x) \wedge P_2(y) \wedge LO_2(x,y)) \vee (P_3(x) \wedge P_3(y) \wedge LO_3(x,y))$

Additional remarks

- Restrictions on the number of first-order variables

- Restrictions on the quatifier prefix of the first-order part

Restrictions on the number of first-order variables

- The counter example we saw (the unmodified one for $MESO$) requires 9 first-order variables (the modifications add first-order variables).
- Let $MESO_2$ be the set of $MESO$ sentences with at most 2 first-order variables.
- The 0-1 law fails for $MESO_2$.
- It is not proved whether the 0-1 law fails or holds for $MESO_2$ on undirected graphs.

Restrictions on the quatifier prefix of the first-order part, I

- Another possibility is to define fragments of ESO (or $MESO$) by considering restrictions on the quantifier prefixes of the first-order part. Some nontrivial restrictions of ESO have the 0-1 law.

- An ESO sentence can be written as $\exists X_1 ... \exists X_n Q_1 x_1 ... Q_m x_m \varphi(X_1, ..., X_m, x_1, ..., x_m)$ where each Q_i is $\forall$ or $\exists$, and φ is quatifier-free.

- If r is a regular expression over the alphabet $\{\exists, \forall\}$, by $ESO(r)$ we denote the set of all sentences s.t. the string $Q_1, ..., Q_m$ is in the language denoted by r.

- Examples of ESO fragments that have the 0-1 law:
  - $ESO(\exists^* \forall^*)$
  - $ESO(\exists^* \forall \exists^*)$

Restrictions on the quatifier prefix of the first-order part, II

- Examples of ESO fragments that don't have the 0-1 law:
  - $ESO(\forall\forall\exists)$
  - $ESO(\forall\exists\forall)$
- We proved the second example above. The sentence we used to prove that the 0-1 law fails for ESO in the Introduction section is in $ESO(\forall\exists\forall)$.
- Actually both of the examples don't have the 0-1 law even if the first-order part does not use equality.

References

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