A Computational Framework for the Study of Partition Functions and Graph Polynomials

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Partition functions and graph polynomials have found many applications in combinatorics, physics, biology and even the mathematics of finance. Studying their complexity poses some problems. To capture the complexity of their combinatorial nature, the Turing model of computation and Valiant's notion of counting complexity classes seem most natural. To capture the algebraic and numeric nature of partition functions as real or complex valued functions, the Blum-Shub-Smale (BSS) model of computation seems more natural. As a result many papers use a naive hybrid approach in discussing their complexity or restrict their considerations to sub-fields of $\mathbb{C}$ which can be coded in a way to allow dealing with Turing computability.

In this paper we propose a unified natural framework for the study of computability and complexity of partition functions and graph polynomials and show how classical results can be cast in this framework.

Keywords: Graph polynomials, partition functions, models of computation, complexity of computation, counting problems

1. Introduction

The study of graph polynomials and partition functions has become a focal point in the research linking discrete mathematics to applied mathematics. Initiated 100 years ago with Birkhoff's paper on the chromatic polynomial, [1], and extended in W. Tutte's paper of 1954, [2], graph polynomials remained for long an exotic subject. However, physicists and chemists independently were led to study similar mathematical abstractions, [3,4]. For recent survey, see [5]. Since the 1980ties many versions of partition functions (aka Potts models, Ising models, Jones polynomial) found applications in fields as diverse as statistical mechanics, chemical graph theory, knot theory, biology and the mathematics of finance.

In their landmark paper [6] F. Jaeger, D.L. Vertigan and D.J.A. Welsh analyzed the complexity of the Tutte and Jones polynomials. Given a point in the complex plane $\overline{a}$ they looked at the complex valued graph parameter $T_{\overline{a}}(-)$ defined by the evaluation $T(-;\overline{a})$ of the Tutte polynomial at the point $\overline{a}$.

The Tutte polynomial is a special case of a *graph polynomial*, a functor which associates with a graph $G = (V, E)$ a polynomial in a polynomial ring $\mathcal{R}$ which is invariant under graph isomorphisms.

Let $\mathcal{R}$ be a sub-field of the complex numbers $\mathbb{C}$. Let $P(G; \overline{X})$ be a graph polynomial in the indeterminates $X_1, \ldots, X_n$ with coefficients in $\mathcal{R}$. For $\overline{a} \in \mathcal{R}^n$, $P(-;\overline{a})$ is a graph invariant taking values in $\mathcal{R}$. We could restrict the graphs to be from a class (graph property) $\mathcal{C}$ of graphs.

We are interested in the complexity of computing $P(-;\overline{a})$ for graphs from $\mathcal{C}$. If for all graphs $G \in \mathcal{C}$ the value of $P(-;\overline{a})$ is a graph invariant taking values in $\mathbb{N}$, we can work in the *Turing model of computation.* Otherwise we identify the graph G with its adjacency matrix M_G, and we work in the *Blum-Shub-Smale (*BSS*) model of computation*, mostly over the complex numbers $\mathbb{C}$.

We take our inspiration from the classical result of F. Jaeger and D.L. Vertigan and D.J.A. Welsh on the complexity of evaluations of the Tutte polynomial, [6]. They show:

- either evaluation at a point $(a, b) \in \mathbb{C}^2$ is polynomial time computable in the Turing model, and (a, b) lies in some quasi-algebraic set of dimension 1,
- or some $\sharp\mathbf{P}$-complete problem is reducible to the evaluation at $(a, b) \in \mathbb{C}^2$.
- To stay in the Turing model of computation, they assume that (a, b) is in some finite dimensional extension of the field $\mathbb{Q}$.

The proof of the second part is *hybrid* in its nature: The reduction is more naturally placed in the BSS *model* of computation, However, although there are various analogues for $\sharp\mathbf{P}$ in BSS, cf. [7–10] there seemed to be no counterpart for $\sharp\mathbf{P}$-completeness in the BSS model suitable for graph polynomials. In [7,8] what is proposed as $\sharp\mathbf{P}$ for BSS counts the zeros of a polynomial, in case this is finite. Nevertheless, we think it *more natural* to work entirely in the BSS model of computation, and we will propose a new framework based on evaluations of polynomials rather than on counting zeros.

This paper is a first step towards a more general theory. First we formulate the situation described above for the Tutte polynomial entirely in the BSS model of computation over the complex numbers $\mathbb{C}$.

We use the framework of SOL-polynomials introduced in [11]. SOL-polynomials form a large class of graph polynomials, which are in a precise sense definable in Second Order Logic. These are then used to define the class $\mathrm{SOLEVAL}_{\mathbb{C}}$ of graph parameters which are evaluations of SOL-polynomials in $\mathbb{C}$. The class $\mathrm{SOLEVAL}_{\mathbb{C}}$ will serve as the previously missing counterpart for $\sharp\mathbf{P}$ in the BSS model. We then investigate to what extent the Tutte polynomial is typical and formulate several versions of "Difficult Evaluation Properties" and "Difficult Point Properties" (DPP). We examine many cases from the literature where these "Difficult Evaluation Properties" hold. Finally, we formulate several conjectures.

The value of the paper is mostly conceptual. It puts complexity questions of graph polynomials into a uniform framework which allows to compare results scattered in the literature. But last but not least, it opens new avenues of research.

We assume the reader is vaguely familiar with the BSS-model of computation, cf. [12], and with the basic of complexity of counting, cf. [13,14].

2. The complexity of graph parameters with values in $\mathbb{C}$

2.1. *Valiant's counting functions and their Turing complexity*

L. Valiant in [15] introduced the counting complexity class $\sharp\mathbf{P}$ which has complete problems with respect to polynomial time Turing reductions. Typical $\sharp\mathbf{P}$-complete problems are the number of 3-colorings of a graph or the number of perfect matchings. Degrees are equivalence classes of decision problems or counting problems with respect to $\mathbf{P}$-time Turing reducibility. Ladner's Theorem asserts that, assuming $\mathbf{P} \neq \mathbf{NP}$, for every degree $[g] \in \mathbf{NP} - \mathbf{P}$ there is $[g'] \in \mathbf{NP} - \mathbf{P}$ with $[g'] < [g]$, [16–18]. It seems to be folklore that the same holds also, if we replace $\mathbf{NP}$ by $\sharp\mathbf{P}$ and $\mathbf{P}$ by $\mathbf{FP}$.

2.2. *Graph parameters in the* BSS*-model*

We define the framework for graph parameters in the BSS model over some sub-field $\mathcal{R}$ of the complex numbers $\mathbb{C}$. When we get more specific $\mathcal{R}$ will be

$\mathbb{C}$. A *graph invariant* or *graph parameter* is a function $f : \bigcup_n \{0,1\}^{n \times n} \to \mathcal{R}$ which is invariant under permutations of columns and rows of the input adjacency matrix. Graph invariants include decision problems. A *graph transformation* is a function $T : \bigcup_n \{0,1\}^{n \times n} \to \bigcup_n \{0,1\}^{n \times n}$ which is invariant under permutations of columns and rows of the input adjacency matrix. The BSS-*P-time computable functions* over $\mathcal{R}$, $\mathrm{FP}_{\mathcal{R}}$ are the functions $f : \{0,1\}^{n \times n} \to \mathcal{R}$ BSS-computable in time $O(n^c)$ for some fixed $c \in \mathbb{N}$. Let f_1, f_2 be graph invariants. f_1 *is* BSS-*P-time reducible to* f_2, $f_1 \leq_P f_2$ if there are BSS-P-time computable functions T and F such that

(i) T is a graph transformation ;
(ii) For all graphs G with adjacency matrix M_G we have

$$f_1(M_G) = F(f_2(T(M_G)))$$

Two graph invariants f_1, f_2 are BSS-*P-time equivalent over* $\mathcal{R}$, $f_1 \sim_{\mathcal{R}} f_2$, if $f_1 \leq_{\mathcal{R}} f_2$ and $f_2 \leq_{\mathcal{R}} f_1$.

2.3. *Cones and degrees*

The BSS model over a ring $\mathcal{R}$ deals traditionally with *decision problems* where the input is an $\mathcal{R}$-vector. A function f maps $\mathcal{R}$-vectors into $\mathcal{R}$. There is a decision problem associated with f: Given $\overline{X}$ is it true that $f(\overline{X}) = a$. In the study of graph polynomials decision problems and functions have as input $(0,1)$-matrices and the decision problems and functions have to be *graph invariants*.

We denote by $\mathrm{FEXP}_{\mathbb{C}}$ the set of functions computable in time $2^{O(n^c)}$ in the BSS-model over $\mathbb{C}$. If $c = 1$ we write $\mathrm{FE}_{\mathbb{C}}$ and speak of simple exponential time. $\mathrm{FP}_{\mathbb{C}}$ are the functions computable in polynomial time. Let g, g' be two graph parameters in $\mathrm{FEXP}_{\mathbb{C}}$. We denote by $[g]_{\mathcal{R}}$ the equivalence class (BSS-degree) of all graph parameters $g' \in EXP_{\mathcal{R}}$ under the equivalence relation $\sim_{\mathcal{R}}$. Analogously, $[g]_T$ denotes the corresponding equivalence class in the Turing model of computations. We denote by $\langle g \rangle_{\mathcal{R}}$ the class (BSS-cone) $\{g' \in \mathrm{FEXP}_{\mathcal{R}} : g \leq_{\mathcal{R}} g'\}$. There are BSS-NP-complete problems for all $\mathcal{R}$ under consideration here, and instead of specifying them, we consider the complete problems in $\mathbf{NP}_{\mathcal{R}}$ to be a degree (which may vary with the choice of the Ring $\mathcal{R}$). The cone of an **NP**-complete problem forms the **NP**-hard problems. There is no well developed theory of degrees and cones for functions, especially for graph parameters, in the BSS model.

G. Malajovich and K. Meer proved an analogue of Ladner's Theorem for decision problems in the BSS-model over $\mathbb{C}$. Iterating their argument, one gets:

Theorem 2.1. (G. Malajovich and K. Meer) *Assume* $\mathbf{P}_{\mathbb{C}} \neq \mathbf{NP}_{\mathbb{C}}$. *Then for every degree* $[g] \in \mathbf{NP}_{\mathbb{C}} - \mathbf{P}_{\mathbb{C}}$ *there is* $[g'] \in \mathbf{NP}_{\mathbb{C}} - \mathbf{P}_{\mathbb{C}}$ *with* $[g'] < [g]$.

Note that the corresponding result over the reals $\mathbb{R}$ or any other field $\mathcal{R}$ is not known to hold, cf. [19].

2.4. *Discrete counting in* BSS

K. Meer in [7] introduced an analogue of $\sharp\mathbf{P}$ for discrete counting in the BSS model over $\mathbb{R}$. In [20] a first attempt of studying graph polynomials in the BSS model is discussed.

The definitions from [7] can easily be extended to the BSS model over $\mathbb{C}$. Let us look at *counting functions*, i.e., functions $f : \mathbb{C}^{\infty} \to \mathbb{N} \cup \{\infty\}$. For a complexity class of functions FC we denote by FC^{count} the class of counting functions in FC. Such a function f is in $\sharp\mathbf{P}_{\mathbb{C}}$ if there exists a polynomial time BSS-machine over $\mathbb{C}$ and a polynomial q such that

$$f(y) = |\{z \in \mathbb{C}^{q(size(y)} : M(y, z) \text{ accepts } \}|$$

It is not difficult to see, cf. [7], that

$$\mathrm{FP}^{count}_{\mathcal{R}} \subseteq \sharp\mathbf{P}_{\mathcal{R}} \subseteq \mathrm{FE}_{\mathcal{R}}$$

for every sub-field $\mathcal{R}$ of $\mathbb{C}$. Typical examples from [7] over the reals $\mathbb{R}$ are counting zeroes of multivariate polynomials of degree at most 4 ($\sharp 4-$FEAS) or counting the number of sign changes of a sequence of real numbers ($\sharp$SC).

Over the complex numbers also the number of k-colorings of a graph is in $\sharp\mathbf{P}_{\mathbb{C}}$ for fixed k. To see this, we associate with a graph $G = (V, E)$ with $V = \{1, \ldots, n\} = [n]$ the following set $\mathcal{E}^k_{color}(G)$ of equations:

(i) $x_i^k - 1 = 0, i \in [n]$
(ii) $\sum_{d=0}^{k-1} x_i^{k-1-d} x_j^d = 0$, for all $(i, j) \in E$.

Clearly, $\mathcal{E}^k_{color}(G)$ has at most k^n many complex solutions. Now, D.A. Beyer in his Ph.D. thesis, [21], observed that

Proposition 2.1. (D. Beyer) *A graph* G *is* k*-colorable iff* $\mathcal{E}^k_{color}(G)$ *has a complex solution, and each solution corresponds exactly to proper* k*-coloring of* G.

This also shows that for fixed k deciding k-colorability is in $\mathbf{NP}_{\mathbb{C}}$. However, it seems unlikely that it is $\mathbf{NP}_{\mathbb{C}}$-hard, because of the very special form of the equations involved. For further discussion, cf. [22].

In S. Margulies' Ph.D. thesis, [23, Chapter 2] the following is shown:

Theorem 2.2. (S. Margulies) *Every decision problem in* **NP** *(in the Turing model) can be encoded as solvability problem of sets of equations over* $\mathbb{C}$.

Using the fact that ♯SAT is ♯**P** complete we get:

Theorem 2.3. *Every function* $f \in \sharp\mathbf{P}$ *(in the Turing model) has an encoding in* $\sharp\mathbf{P}_{\mathbb{C}}$ *(in the* BSS *model).*

In particular we can place the permanent and Hamiltonian functions in the BSS model over $\mathbb{C}$. These functions are usually studied in Valiant's theory of algebraic circuits, cf. [24].

Corollary 2.1. *The functions* per *and* ham *of* $(0,1)$ *matrices are in* $\sharp\mathbf{P}_{\mathbb{C}}$.

2.5. *The difficult counting hypothesis (DCH)*

There are very few explicit $\sharp\mathbf{P}_{\mathbb{C}}$-complete problems in the literature. The paper [25] shows that the computation of the Euler characteristic of an affine or projective complex variety is complete in this class for Turing reductions in the BSS-model of computation. But there are no explicit $\sharp\mathbf{P}_{\mathbb{C}}$-complete problems in graph theory which correspond to problems which are in ♯**P**-complete problems in the Turing model of computation. This is due to the fact that counting discrete solution sets of polynomial equations does not correspond to solvability in a parsimonious way. However, some problems in $\sharp\mathbf{P}_{\mathbb{C}}$ are $\mathbf{NP}_{\mathbb{C}}$-hard, because a set of polynomial equations is solvable if the number of solutions in the above sense is different from 0 but may be ∞. Solvability of systems of polynomial equations is $\mathbf{NP}_{\mathbb{C}}$-complete. It might not be too difficult to construct artificilly problems which are $\sharp\mathbf{P}_{\mathbb{C}}$-complete.

However, k-colorability as expressed by a set of equations is *unlikely* to be $\mathbf{NP}_{\mathbb{C}}$-hard, because of the special form of the equations. Therefore, also counting the number of colorings is not known to be $\mathbf{NP}_{\mathbb{C}}$-hard in the BSS-model. On the other hand, it would be *truly surprising* if counting the number of k-colorings were in $\mathrm{FP}_{\mathbb{C}}^{count}$.

We therefore formulate the following two complexity hypotheses for the BSS model over $\mathbb{C}$:

Strong difficult counting hypothesis (SDCH)
Every counting function in $f \in \sharp\mathbf{P}_{\mathbb{C}}$ with discrete input which is $\sharp\mathbf{P}$-hard in the Turing model is $\mathbf{NP}_{\mathbb{C}}$-hard in the BSS model over $\mathbb{C}$.

and

Weak difficult counting hypothesis (WDCH)
A counting function in $f \in \sharp\mathbf{P}_{\mathbb{C}}$ which is $\sharp\mathbf{P}$-hard in the Turing model cannot be in $\mathrm{FP}_{\mathbb{C}}^{count}$.

The following is easy to see:

Proposition 2.2. *Assume $\mathbf{P}_{\mathbb{C}} \neq \mathbf{NP}_{\mathbb{C}}$. Then **SDCH** implies **WDCH**.*

2.6. *Evaluations of graph polynomials over* $\mathbb{C}$

Let $\mathcal{C}$ be a graph property. Let $P(G; \overline{X})$ be a graph polynomial with indeterminates $X_1, \ldots, X_m$. We define

$$\mathrm{EASY}_{\mathbb{C}}(P, \mathcal{C}) = \{\overline{a} \in \mathbb{C}^n : P(-; \overline{a}) \in \mathrm{FP}_{\mathbb{C}}\}$$

and

$$\mathrm{HARD}_{\mathbb{C}}(P, \mathcal{C}) = \{\overline{a} \in \mathbb{C}^n : P(-; \overline{a}) \text{ is } \mathbf{NP}_{\mathbb{C}} - \text{hard}\}$$

Let f be a counting function not in $\mathrm{FP}_{\mathbb{C}}^{count}$ or a decision problem not in $\mathbf{P}_{\mathbb{C}}$.

$$f - \mathrm{HARD}_{\mathbb{C}}(P, \mathcal{C}) = \{\overline{a} \in \mathbb{C}^n : P(-; \overline{a}) \in \langle f \rangle_{\mathbb{C}}\}$$

We omit $\mathcal{C}$ if $\mathcal{C}$ is the class of all finite graphs.

Clearly, if **SDCH** is true and f is $\sharp\mathbf{P}$-complete in the Turing model then

$$f - \mathrm{HARD}_{\mathbb{C}}(P, \mathcal{C}) = \mathrm{HARD}_{\mathbb{C}}(P, \mathcal{C}).$$

How can we describe $\mathrm{EASY}_{\mathbb{C}}(P, \mathcal{C})$ and $\mathrm{HARD}_{\mathbb{C}}(P, \mathcal{C})$?

3. Case studies: Three graph polynomials

3.1. *The chromatic polynomial and its complexity*

Let $G = (V(G), E(G))$ be a graph, and $\lambda \in \mathbb{N}$.

A *proper λ-vertex-coloring* is a map $c : V(G) \to [\lambda]$ such that $(u, v) \in E(G)$ implies that $c(u) \neq c(v)$. Let $\chi(G, \lambda)$ be the number of proper λ-vertex-colorings. Hundred years ago in 1912, G. Birkhoff showed that $\chi(G, \lambda)$ is a *polynomial in* $\mathbb{Z}[\lambda]$. Henceforth we treat $\chi(G, \lambda)$ as a polynomial over $\mathbb{C}$ and consider also evaluations of λ for arbitrary complex numbers. $\chi(G,\lambda)$ is called the *chromatic polynomial of* G. In 1973, R. Stanley showed that for simple graphs G, $\mid \chi(G,-1) \mid$ counts the number of *acyclic orientations* of G. These classical results can be found in many monographs, e.g., [26–28].

For fixed $a \in \mathbb{C}$ we define the graph parameter $\chi_a(G)$ and look at its complexity as a function of G. We follow [29]. There are three values $a = 0, 1, 2$ for which $\chi_a(G)$ can be computed in polynomial time both in the Turing model and in the BSS-model of computation. Furthermore, for $\chi_3(G)$ and $\chi_{-1}(G)$ are $\sharp\mathbf{P}$-complete in the Turing model of computation.

Let $G_1 \bowtie G_2$ denote the join of two graphs. We observe that

$$\chi(G \bowtie K_n, \lambda) = (\lambda)^{\underline{n}} \cdot \chi(G, \lambda - n) \qquad (\star)$$

From this we get

(i) $\chi(G \bowtie K_1, 4) = 4 \cdot \chi(G, 3)$
(ii) $\chi(G \bowtie K_n, 3+n) = (n+3)^{\underline{n}} \cdot \chi(G, 3)$, hence for $n \in \mathbb{N}$ with $n \geq 3$ it is $\sharp\mathbf{P}$-complete.

Here $x^{\underline{n}} = x \cdot (x-1) \cdot \ldots \cdot (x-n+1)$. These reductions work in the Turing model for λ in some Turing-computable field extending $\mathbb{Q}$. The reductions also work in BSS if performed directly on the graph parameters, and not on the equivalent problem of solvability of the equations $\mathcal{E}^k_{color}(G)$.

If we have an oracle for some $q \in \mathbb{Q} - \mathbb{N}$ which allows us to compute $\chi_q(G)$ we can compute $\chi(G, q')$ for any $q' \in \mathbb{Q}$ as follows:

Algorithm $A(q, q', \mid V(G) \mid)$**:**

(i) Given G the degree of $\chi(G, q)$ is at most $n = \mid V(G) \mid$.
(ii) Use the oracle and $(\star)$ to compute $n+1$ values of $\chi(G, \lambda)$.
(iii) Using Lagrange interpolation we can compute $\chi(G, q')$ in polynomial time.

We note that this algorithm is purely algebraic and works for all graphs G, $q \in (F) - \mathbb{N}$ and $q' \in F$ for any field F extending $\mathbb{Q}$.

Hence we get that for all $q_1, q_2 \in \mathbb{C} - \mathbb{N}$ the graph parameters are *polynomially reducible to each other*.

Furthermore, for $3 \leq i \leq j \in \mathbb{N}$, $\chi(G, i)$ is reducible to $\chi(G, j)$. This now works in the BSS-model over $\mathbb{C}$.

3.2. *Dichotomy of the difficulty of evaluations*

We summarize the situation for the chromatic polynomial as follows:

(i) $\mathrm{EASY}_{\mathbb{C}}(\chi) = \{0, 1, 2\}$
(ii) The remaining cases can be divided into those points $a \in \mathbb{C}$ where $\chi_a(G)$ is both in $\sharp\mathbf{P}$ and $\sharp\mathbf{P}_{\mathbb{C}}$, and into those points a for which there is no counting interpretation.
(iii) In the Turing model the degrees of $\chi_3(G)$ and $\chi_{-1}(G)$ are the same, because both are in $\sharp\mathbf{P}$ and they are $\sharp\mathbf{P}$-complete.
(iv) In BSS over $\mathbb{C}$ we only get that $[\chi_3] \leq [\chi_{-1}]$
(v) If we had $[\chi_3] = [\chi_{-1}]$ then χ_3HARD would consist of all the χ_a with $a \neq 0, 1, 2$. Under SDCH these would indeed be $\mathbf{NP}_{\mathbb{C}}$-hard. Under WDCH it would be different from $\mathrm{EASY}_{\mathbb{C}}(\chi)$.
(vi) Among the graph parameters $\chi_a(G) : a \in \mathbb{C}$, either all are in $\mathrm{EASY}_{\mathbb{C}}(\chi)$ or there is smallest degree, namely $[\chi_3(G)]$ which is not easy. In other words, Ladner's theorem does not hold for the evaluations of the chromatic polynomial, and, assuming that $[\chi_3(G)] \not\in \mathrm{FP}_{\mathbb{C}}$, $[\chi_3(G)]$ is a minimal degree.
(vii) It is conceivable that in BSS over $\mathbb{C}$ we have

$$[\chi_3(G)] < [\chi_4(G)] < \ldots < [\chi_j(G)] < \ldots < [\chi_{-1}(G)] = [\chi_a(G) : a \in \mathbb{C} - \mathbb{N}]$$

We have a *Dichotomy Theorem* for the evaluations of $\chi(-, \lambda)$:

(i) $\mathrm{EASY}_{\mathbb{C}}(\chi) = \{0, 1, 2\}$ is a *quasi-algebraic set* (a finite Boolean combination of algebraic sets) of *dimension* 0.
(ii) All other evaluations are at least as difficult as $\chi_3(G)$, which is a *quasi-algebraic* set of *dimension* 1.
(iii) Under the assumption of **SDCH** we get the dichotomy that all evaluations of $\chi(G, x)$ are either in $\mathrm{EASY}_{\mathbb{C}}(\chi)$ or in $\mathrm{HARD}_{\mathbb{C}}(\chi)$.
(iv) Under the assumption of **WDCH** we get the dichotomy that all evaluations of $\chi(G, x)$ are either in $\mathrm{EASY}_{\mathbb{C}}(\chi)$ or in $\chi_3 - \mathrm{HARD}_{\mathbb{C}}(\chi)$.

3.3. *The complexity of the Tutte and the cover polynomial*

The Tutte polynomial $T(G, X, Y)$ is a bivariate polynomial and $\chi(G, \lambda) \leq_P T(G, 1 - \lambda, 0)$. For our discussion here the exact definition of the Tutte polynomial is not needed. A good reference is [30, Chapter 10].

We have the following dichotomy theorem:

(i) $\mathrm{EASY}_{\mathbb{C}}(T) = \mathrm{H} \cup \mathrm{Except}$, where $H = \{(x, y) \in \mathbb{C}^2 : (x - 1)(y - 1) = 1\}$ and Except contains the points $(0, 0), (1, 1), (-1, -1), (0, -1)$, $(-1, 0), (i, -i), (-i, i)$, $(j, j^2), (j^2, j)$ where $j = e^{\frac{2\pi i}{3}}$. This is a quasi-algebraic set of dimension 1.

(ii) For all other points $a \in \mathbb{C}^2$ evaluating the Tutte polynomial is at least as hard as $T(G, -2, 0)$. This is a quasi-algebraic set of dimension 2.
(iii) Furthermore there is a most difficult evaluation point in $\overline{A}_{hard} \in \mathbb{C}^2$ and most evaluation points are in the degree of $\overline{A}_{hard}$.
(iv) As in the case of the chromatic polynomial, one can use the **WDCH** or **SDCH** to sharpen the dichotomy.

The cover polynomial $C(D; X, Y)$ is a bivariate graph polynomial for digraphs D which was introduced by F.R.K. Chung and R.L. Graham, [31]. Its complexity was studied in [32] and again follows the same pattern.

(i) $\mathrm{EASY}_{\mathbb{C}}(C) = \{(0,0), (0,-1), (1,-1)\}$. This is a quasi-algebraic set of dimension 0.
(ii) For all other points $a \in \mathbb{C}^2$ evaluating the cover polynomial is at least as hard as $C(D; 0, 1)$ or $C(D; 1, 0)$. This is a quasi-algebraic set of dimension 2.
(iii) Again there is also a hardest evaluation point.

Note that $C(D, 0, 1)$ is the permanent of the adjacency matrix of D and $C(D, 1, 0)$ counts the number of Hamiltonian paths of D. Both these evaluations are $\sharp\mathbf{P}$-complete in the Turing model. At the moment it is not clear to us which of them is reducible to the other in BSS over $\mathbb{C}$.

3.4. *Lessons learned from the case study*

In the introduction we proposed to study the complexity of graph parameters in the framework of BSS. Graph parameters are often integer valued, but there are many cases which may have values in $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$, in particular graph parameters of weighted graphs. To include all the cases from the literature we chose to deal with case of complex valued graph parameters. In particular our graph parameters are functions in $\mathrm{FEXP}_{\mathbb{C}}$. We adapted the notions of **P**-time reducibility of BSS to functions in $\mathrm{FEXP}_{\mathbb{C}}$ and to graph parameters in particular. The degree structure of $\mathrm{FEXP}_{\mathbb{C}}$ under **P**-time reducibility is not well understood. At the lowest end we have the functions $\mathrm{FP}_{\mathbb{C}}$. A typical $\mathbf{NP}_{\mathbb{C}}$-hard problem is the solvability of polynomial equations, which can be coded as a graph parameter of a hyper-edge weighted hyper-graph which reflects the structure of the polynomial equations. Therefore some graph parameters are $\mathbf{NP}_{\mathbb{C}}$-hard. However, the exact complexity of most graph parameters in the BSS model over $\mathbb{C}$ is not known, even under the Difficult Counting Hypotheses **SDCH** and **WDCH**.

We have shown that $\sharp\mathbf{P}_{\mathbb{C}}$ does capture surprisingly well the classical counting problems ♯SAT and the like. However, it has the disadvantage of allowing only values in $\mathbb{N}$ or the value ∞. What we need is an extension of $\sharp\mathbf{P}$ of the Turing model in BSS over $\mathbb{C}$ which takes arbitrary values in $\mathbb{C}$. S. Toda's Theorem states that decision problems in $\mathbf{P}^{\sharp\mathbf{P}}$ contain the whole polynomial hierarchy **PH**, cf. [14], or equivalently all the decision problems definable in Second Order Logic, cf. [33]. These consideration lead us to the introduction of the class $\mathrm{SOLEVAL}_{\mathbb{C}}$ of graph parameters as a substitute for $\sharp\mathbf{P}$ of the Turing model.

4. The complexity class SOLEVAL

The class of functions in $\sharp\mathbf{P}_{\mathbb{C}}$ has the disadvantage that it counts only the number of solutions of polynomial equations provided this number is finite. Evaluating the chromatic polynomial at an irrational point is therefore not expected to be in $\sharp\mathbf{P}_{\mathbb{C}}$. We now propose a complexity class for graph parameters, $\mathrm{SOLEVAL}_{\mathcal{R}}$ which is better suited to study the complexity of graph parameters with real or complex values, than generalizations of counting complexity classes.

Our class $\mathrm{SOLEVAL}_{\mathcal{R}}$ contains virtually all graph parameters from standard graph theory books. But it is not difficult to come up with natural candidates for counting problems not in $\mathrm{SOLEVAL}_{\mathcal{R}}$: The game of HEX played on graphs with two disjoint unary predicates on vertices comes to mind. Deciding winning positions is **PSpace**-complete, cf. [34,35]. Therefore counting the winning strategies is not in SOLEVAL unless **PSpace** coincides with the polynomial hierarchy in the Turing model.

In [11] the class of graph parameters definable in Second Order Logic (SOL) was introduced. Roughly speaking these are graph polynomials where summation and products are allowed to range over first order or second order variables of formulas in SOL. For the full definition we refer to [11] but we give a few illustrative examples.

Example 4.1. ***(The independence polynomial)*** Let $ind(G,i)$ denote the number of independent sets of size i of a graph G. The graph polynomial $ind(G,X) = \sum_i ind(G,i) \cdot X^i$, can be written also as

$$ind(G,X) = \sum_{I \subseteq V(G)} \prod_{v \in I} X$$

where I ranges over all independent sets of G and v ranges over all elements in I. To be an independent set is definable by a formula of SOL in the vocabulary of graphs.

Example 4.2. ***(The chromatic number)*** The chromatic number $\chi(G)$ of a graph G is the smallest $k \in \mathbb{N}$ such that $\chi(G; k) \neq 0$. We can represent $\chi(G)$ as the evaluation of the polynomial $\sum_{v \in C : \phi(v,C)} X$ where $\phi(v, C)$ is the formula

$$C(v) \wedge \psi_{coloring}(C) \wedge (\forall C'(\psi_{coloring}(C') \rightarrow \psi_{inject}(C, C')))$$

where

(i) $\psi_{coloring}(C)$ says that "there is a function $f : V(G) \rightarrow C$ which is a proper coloring".
(ii) $\psi_{inject}(C, C')$ says that "there is an injective function $s : C \rightarrow C'$"

Here the vocabulary of graphs is augmented by a unary predicate C, but the polynomial is independent of the the interpretation of C.

A *simple* SOL-*polynomial* $p(G, X)$ is a polynomial of the form

$$p(G, X) = \sum_{A : A \subseteq V(G) : \phi(A)} \prod_{v : \psi(v)} X$$

where A ranges over all subsets of $V(G)$ satisfying $\phi(A)$ and v ranges over all elements of $V(G)$ satisfying ψ. Both formulas ϕ and ψ are SOL-formulas, but summation be be over first and second order variables, while products are only over first order variables.

For the general case

- One allows several indeterminates $X_1, \ldots, X_t$.
- One allows summation over relations $S \subseteq V(G)^k$ of any fixed arity k rather than over sets.
- Instead of the standard monomials other bases of monomials can be used. For example, in the case of one indeterminate X^n could be replace by $\binom{X}{n}$ or the falling factorials $X^{\underline{n}}$.
- One gives an inductive definition.
- One allows arbitrary finite relational vocabularies and is not restricted to the vocabulary of graphs.
- In some applications one looks at graphs with a linear order on the vertices, but one requires the definition to be *invariant under the ordering*, i.e., different orderings still give the same polynomial.

The general case includes the chromatic polynomial, the Tutte polynomial and its variations, the cover polynomial, and virtually all graph polynomials from the literature. We shall see more examples in the sequel.

Proposition 4.1. *For every field $\mathcal{R}$ we have*

(i) $\mathrm{SOLEVAL}_{\mathcal{R}} \subseteq \mathrm{FEXP}_{\mathcal{R}}$.
(ii) If $f \in \sharp\mathbf{P}$ *in the Turing model, then* $f \in \mathrm{SOLEVAL}_{\mathcal{R}}$.

Sketch. (i) is shown by induction on the defining formulas. (ii) follows from Fagin's characterization of **NP** in the Turing model as the problems definable in existential SOL, cf. [33]. □

5. Degrees of evaluations

Ultimately one wants to study the structure of the degrees in $\mathrm{SOLEVAL}_{\mathcal{R}}$ for various sub-fields of $\mathbb{C}$. Is there a maximal degree in $\mathrm{SOLEVAL}_{\mathcal{R}}$, in other words, does it have complete graph parameters? Does it contain interesting sub-hierarchies?

Here we set ourselves a more moderate task. We start a general investigation of the degree structure of the evaluations of a fixed (set of) graph polynomial(s).

Let $\mathfrak{P}$ be a family of SOL-polynomials. If $\mathfrak{P} = \{F\}$ is a singleton, we omit the set brackets. For a sub-field $\mathcal{R} \subseteq \mathbb{C}$ we denote by $\mathrm{EVAL}_{\mathcal{R}}(\mathfrak{P})$ the set of graph parameters which are evaluations of polynomials in $\mathfrak{P}$ over $\mathcal{R}$. If $\mathfrak{P}$ consists of all SOL-polynomials, we denote it by $\mathrm{SOLEVAL}_{\mathcal{R}}$.

5.1. *The partial order of the degrees*

If $F(G : \overline{X})$ is a a SOL-polynomial in n indeterminates $\overline{X} = (X_1, \ldots, X_n)$ we can partially pre-order its evaluations points $\overline{a} \in \mathbb{C}^n$ by the partial pre-order of the degrees of evaluating $F(G, \overline{a})$. So we write $[\overline{a}]_F$ for the degree of $F(G, \overline{a})$. What interest us is the partial order one gets by taking the quotient of the pre-order with respect to the equivalence relation of polynomial time bi-reducibility.

Example 5.1. The quotient order on $\mathrm{EVAL}(\chi)$ of the chromatic polynomial is discrete and linear with $\mathrm{FP}_{\mathbb{C}}$ as its first element and $[\chi(G : -1)]$ as its last element.

Example 5.2. The quotient order on $\mathrm{EVAL}(T)$ of the Tutte polynomial consists of one element, or it has a minimal element above $\mathrm{FP}_{\mathbb{C}}$ and a maximal element. The case of the cover polynomial is similar.

5.2. *Difficult points are dense*

Another way of studying the complexity of $\mathrm{EVAL}(F)$ for a particular SOL-polynomial in n indeterminates is topological.

Theorem 5.1. *Let $\mathcal{R}$ be a sub-field of $\mathbb{C}$ which is a metric space. Let $F(G;\overline{X})$ be a* SOL*-polynomial such that for some $\overline{a} \in \mathbb{C}^n$ the graph parameter $F(G;\overline{a}) \notin \mathrm{FP}_{\mathcal{R}}$. Then the set of difficult evaluation points $\mathrm{EVAL}(F)_{\mathcal{R}} - \mathrm{FP}_{\mathcal{R}}$ is dense in $\mathcal{R}$.*

Sketch. Assume there is a neighborhood U where all evaluation points are in $\mathrm{FP}_{\mathcal{R}}$. So we can compute in polynomial time enough values of $F(G,x)$ to use multi-dimensional Lagrange interpolation and compute the coefficients of $F(G,x)$. But then all evaluations of $F(G,a)$ are in $\mathrm{FP}_{\mathcal{R}}$. □

In the examples so far more is true:

Example 5.3. The evaluations of the maximal degree of $\mathrm{EVAL}(\chi)_{\mathcal{R}}$ are dense in $\mathcal{R}$.

Example 5.4. The evaluations of the maximal degree of $\mathrm{EVAL}(T)_{\mathcal{R}}$ are dense in $\mathcal{R}^2$.

Theorem 5.2. *Let F be a* SOL*-polynomial in n indeterminates. Assume there is an open set $U \subseteq \mathbb{C}^n$ and a point $\overline{A} \in \mathbb{C}^n$ such that for all $\overline{b} \in U$ the degree $[\overline{b}]_F \leq [\overline{a}]_F$. Then $[\overline{a}]_F$ is a maximal degree in* $\mathrm{EVAL}(F)_{\mathbb{C}}$.

Sketch. The previous proof still works, since $[\overline{a}]_F$ is closed under polynomial reductions and computations. □

6. The difficult point property

Given a graph polynomial $F(G,\overline{X})$ in n indeterminates $X_1,\dots,X_n$ we are interested in the set $\mathrm{EASY}_{\mathbb{C}}(F)$.

We say that F has the *weak difficult point property* (**WDPP**) if

(i) there is a quasi-algebraic subset $A \subset \mathbb{C}^n$ of dimension $\leq n-1$ which contains $\mathrm{EASY}_{\mathbb{C}}(F)$, and
(ii) there exists finitely many $\overline{a}_i : i \leq \alpha \in \mathbb{N}$ such that for each $\overline{b} \in \mathbb{C}^n - A$ the evaluation $F(G,\overline{b})$ is $F(G,\overline{a}_i) - \mathrm{HARD}_{\mathbb{C}}$ for some $\overline{a}_i \in \mathbb{C}^n$ and for all $i \leq \alpha$ the evaluation $F(G,\overline{a}_i)$ is not in $\mathrm{FP}_{\mathbb{C}}$.

We say that F has the *strong difficult point property* (**SDPP**) if **WDPP** holds for F with $A = \mathrm{EASY}_{\mathbb{C}}(F)$.

Without the requirement that A has a small dimension the **SDPP** is a *dichotomy property*, in the sense that every evaluation point is either easy or at least as hard as one of the evaluations at $\overline{a}_i, i \leq \alpha$. The two versions,

WDPP and **SDPP**, have a *quantitative aspect*: The set of easy points is rare in a strong sense: they are in a quasi-algebraic set of lower dimension.

Again the Difficult Counting Hypotheses **WDCH** and **SDCH** can be used to sharpen both Difficult Point Properties.

7. Examples for the WDPP and SDPP

In the discussion of the examples in this section we assume the Weak Difficult Counting Hypothesis **WDCa**H. We have seen in the case study, that the chromatic polynomial $\chi(G;\lambda)$ and the Tutte polynomial $T(G;X,Y)$ both have the **SDPP** with $\alpha = 1$ and the cover polynomial $C(D;X,Y)$ has the **SDPP** with $\alpha = 2$.

7.1. *The Bollobás-Riordan polynomials*

The Bollobás-Riordan polynomial is a generalization of the Tutte polynomial for graphs with colored edges colored with k colors, [36]. It has $4k$ many indeterminates. Its complexity was studied in [37,38], where it was shown that it satisfies **WDPP**.

7.2. *The interlace polynomials*

The interlace polynomial was introduced and intensively studied in [39–43]. It is a polynomial in two indeterminates. It complexity was studied in [44,45] where it was shown that it satisfies the **WDPP**.

7.3. *Counting weighted homomorphisms aka partition functions*

Let $A \in \mathbb{C}^{n \times n}$ be a complex, symmetric matrix, and let G be a graph. Let

$$Z_A(G) = \sum_{\sigma: V(G) \to [n]} \prod_{(v,w) \in E(G)} A_{\sigma(v),\sigma(w)}$$

Z_A is called a *partition function*.

If $\mathbf{X}$ is the matrix $(X_{i,j})_{i,j \leq n}$ of indeterminates, then $Z_{\mathbf{X}}$ is a graph polynomial in n^2 indeterminates, and Z_A is an evaluation of $Z_{\mathbf{X}}$. i

Partition functions have their origin in statistical mechanics and have a very rich literature. In [46] a characterization is given of all multiplicative graph parameters which can be presented as partition functions. The complexity of partition functions was studied in a series of papers, [47–50].

Jin-yi Cai, Xi Chen and Pinyan Lu, [50], building on [47] proved a dichotomy theorem for $Z_{\mathbf{X}}$ where $\mathcal{R} = \mathbb{C}$. Analyzing their proofs reveals that $Z_{\mathbf{X}}$ satisfies the **SDPP** for $\mathcal{R} = \mathbb{C}$.

There are various generalizations of this to Hermitian matrices, cf. [51], and beyond.

7.4. *Generalized colorings*

Let $f : V(G) \to [k]$ be a coloring of the vertices of $G = (V(G), E(G))$. We look at variations of coloring properties which have been defined in the literature, cf. [11].

(i) f is *proper* if $(uv) \in E(G)$ implies that $f(u) \neq f(v)$. In other words if for every $i \in [k]$ the counter-image $[f^{-1}(i)]$ induces an independent set. These are the usual colorings.
(ii) f is *convex* if for every $i \in [k]$ the counter-image $[f^{-1}(i)]$ induces a connected graph, cf. [52].
(iii) f is *t-improper* if for every $i \in [k]$ the counter-image $[f^{-1}(i)]$ induces a graph of maximal degree t. For its origins and history cf. [53].
(iv) f is *H-free* if for every $i \in [k]$ the counter-image $[f^{-1}(i)]$ induces an H-free graph. For its origins and history cf. [54].
(v) f is *acyclic* if for every $i, j \in [k]$ the union $[f^{-1}(i)] \cup [f^{-1}(i)]$ induces an acyclic graph, [55].

The following was shown in [11,56]:

Theorem 7.1. *For all the above properties, counting the number of colorings is a polynomial in k.*

7.5. *More cases where SDPP holds*

SDPP was verified for

(i) the cover polynomial $C(G; x, y)$ introduced in [31] in [57,58].
(ii) the bivariate matching polynomial for multi-graphs defined first in [4] in [59,60].
(iii) The first two authors have also verified it for the graph polynomials for convex colorings, for t-improper colorings (for multi-graphs), for acyclic colorings, and the bivariate chromatic polynomial introduced by K. Döhmen, A. Pönitz and P. Tittman in [61].
(iv) More cases can be found in the Ph.D. Theses of I. Averbouch and the first author [62,63] and in [64,65].

C. Hoffmann's PhD thesis [66] contains a general *sufficient criterion* which allows to establish the **WDPP** for a wide class of (artificially defined) graph polynomials. Unfortunately, this method does not apply in most concrete cases of generalized chromatic polynomials discovered using the methods of [11].

Problem 7.1. *Characterize the graphs H for which the evaluation of the graph polynomial of H-free colorings with k colors satisfies **SDPP** or **WDPP**.*

8. Conclusion, conjectures and open problems

The purpose of this paper was to promote the study of graph parameters in BSS over $\mathbb{C}$ by formulating and illustrating the framework and formulating some conjectures. In particular we want to renew interest in Meer's approach to counting problems over some field. In [7,67,68] the authors studied definability questions but did not study complexity and the degrees of polynomial time reducibility.

It turns out that Meer's definition of $\sharp\mathbf{P}$ in BSS adapted to the complex case, is richer than originally assumed. By translating graph properties into polynomial equations we showed that every problem in $\sharp\mathbf{P}$ in the Turing model is also in $\sharp\mathbf{P}_{\mathbb{C}}$. However, the degree structure of $\sharp\mathbf{P}_{\mathbb{C}}$ remains unclear.

Problem 8.1. *Does $\sharp\mathbf{P}_{\mathbb{C}}$ have complete problems?*

Even the complexity of classically difficult problems seems unresolved. For example it is not clear whether 3-colorability is $\mathbf{NP}_{\mathbb{C}}$-hard or whether evaluating the chromatic polynomial at the point -1 is really more difficult than evaluating at the point 3. Similarly, it is not known whether counting the number of perfect matchings or of Hamiltonian paths are of the same difficulty or even comparable. The latter is particularly unsettling, since in Valiant's theory of algebraic circuits computing the Hamiltonian of a matrix and computing the permanent are of the same difficulty (namely **VNP**-complete).

Problem 8.2. *What can we say in* BSS *over $\mathbb{C}$ about the four graph parameters*

(i) $\chi_3(G)$, $\chi_{-1}(G)$,
(ii) $\sharp\mathrm{pm}(G)$, *which counts the number of perfect matchings, and*
(iii) $\sharp\mathrm{ham}(G)$ *which counts the number of Hamiltonian paths?*

Are they $\mathbf{NP}_{\mathbb{C}}$ *hard? Are they mutually equally hard?*

As a more suitable complexity class for the study of graph polynomials and graph parameters we introduced the class $\mathrm{SOLEVAL}_{\mathbb{C}}$.

Problem 8.3. *Does* $\mathrm{SOLEVAL}_{\mathbb{C}}$ *have complete problems? More generally what is the structure of the* $\mathbf{P}_{\mathbb{C}}$*-degrees of* $\mathrm{SOLEVAL}_{\mathbb{C}}$*?*

By translating the results on the complexity of the Tutte polynomial into the BSS model over $\mathbb{C}$ we were able to identify the open problems the solution of which are needed to draw a complete picture.

Let us conclude with a few conjectures in BSS over $\mathbb{C}$:

Conjecture 8.1. (Difficulty of Counting)

(i) ***WDCH*** *is true.*
(ii) ***SDCH*** *is false*

Conjecture 8.2. (Difficulty Point Property)
For every SOL *polynomial* F *in* n *indeterminates the following holds:*

(i) *There is a maximal degree* $[a]_{max}$ *in* $\mathrm{EVAL}(F)_{\mathbb{C}}$. *Furthermore, the evaluation points in* $[a]_{max}$ *form a quasi-algebraic set of dimension* n.
(ii) *Either* $\mathrm{EVAL}(F)_{\mathbb{C}} - \mathrm{FP}_{\mathbb{C}} = \emptyset$ *or it has a minimal degree.*
(iii) $\mathrm{EVAL}(F)_{\mathbb{C}} \cap \mathrm{FP}_{\mathbb{C}}$ *is quasi-algebraic of dimension* $\leq n-1$.

This conjecture is stronger that **SDPP**. In **SDPP** we stipulate that there are finitely many minimal degrees in $\mathrm{EVAL}(F)_{\mathbb{C}} - \mathrm{FP}_C$. In Conjecture 8.2 we require that there is only one such degree. The Strong Difficult Counting Hypothesis **SDCH** would imply that this minimal degree is at least $\mathbf{NP}_{\mathbb{C}}$-hard. We could actually strengthen Conjecture 8.2 by requiring that there is exactly one degree in $\mathrm{EVAL}(F)_{\mathbb{C}} - \mathrm{FP}_C$. But in the light of conjecture 8.1 this seems like going too far.

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