Non-Definability in First Order Logic and Monadic Second Order Logic

Ehrenfeucht-Fraïssé Games

Tools to Show Non-Definability

- Ehrenfeucht-Fraïssé Games
- Translation Schemes and transductions
- Feferman-Vaught Theorem for sums
- $0-1$ Laws

Proving non-definability

The class of τ-structures of finite even cardinality, $EVEN(\tau)$, is *not definable* in First Order Logic, (not even in Monadic Second Order Logic):

- For FOL: use compactness. Every formula true in all finite even structures has an infinite model.

- For FOL (restricted to finite structures): use Pebble Games (Ehrenfeucht-Fraïssé Games)

- For $MSOL$: use Pebble Games adapted to $MSOL$.

Similarly, $DisPath(n)$ is not FOL-definable even for $n = 1$.

Compactness of FOL

Recall:
Σ is satisfiable if there is a τ-structure $\mathcal{A}$ such that $\mathcal{A} \models \Sigma$.

Theorem:[Gödel-Mal'cev]
Let Σ be an infinite set of $FOL(\tau)$-sentences. Σ is satisfiable iff every finite subset $\Sigma_0 \subseteq \Sigma$ is satisfiable.

This theorem was stated and proved in
Logic for CS *for Propositional Logic.*
This theorem was stated, but probably not proved in
Logic for CS *for First Order Logic.*
The proof for FOL is very similar to the one for Propositional Logic.

Using Compactness

Let ϕ_n be the sentence which says that the universe contains at least n elements.

Let Σ_{even} consist of

$$\{\left(\phi_{2n+1} \to \phi_{2n+2}\right) : n \in \mathbb{N}\}$$

All **finite** models of Σ_{even} are of even cardinality.

Assume there is ψ_{even} such that

$$\mathcal{A} \models \psi_{even} \text{ iff } \mid A \mid = 2n$$

Define

$$\Sigma_1 = \{\psi_{even}\} \cup \{\phi_n : n \in \mathbb{N}\} \cup$$

Every finite subset $\Sigma_0 \subseteq \Sigma_1$ is satisfiable (by a finite model of even cardinality).

But Σ_1 has no model, contradicting compactness.

$MSOL$ is not compact

Let $\tau_{a,b} = \tau_{graph} \cup \{a, b\}$ be the vocabulary of graphs with two constants.

In $MSOL(\tau_{a,b})$ we have a formula ϕ_{conn} which says that the graph is connected.

Let $\psi_n(a, b)$ say that the shortest path between a, b is of length n. This is in $FOL(\tau_{a,b})$.

Now every finite subset of

$$\Sigma = \{\phi_{conn} \cup \{\psi_n(a, b) : n \in \mathbb{N}\}$$

is satisfiable, but Σ is not.

Quantifier rank of a formula, I

We write a formula ϕ as a tree:

$$\exists X_1 \forall x_2 \, (x_2 \in X_1 \rightarrow \exists x_3 E(x_2, x_3))$$

$$\exists X_1$$

$$\forall x_2$$

$$\rightarrow$$

$$x_2 \in X_1 \qquad\qquad\qquad \exists x_3$$

$$E(x_2, x_3)$$

The quantifier rank is biggest number of quantifiers one can find along a path in this tree. Here it is 3.

Quantifier rank of a formula, II

- For formulas in prenex normal form the quantifier rank equals the number of quantifiers.

- If we reuse variables, the quantifier rank can be smaller than the number of quantifiers used in prenex normal form.

$$\forall x_1 \, (\exists x_2 E(x_1, x_2) \wedge \exists x_2 \neg E(x_1, x_2))$$

Quantifier rank 2

$$\forall x_1 \exists x_2 \exists x_3 \, (E(x_1, x_2) \wedge \neg E(x_1, x_3))$$

Quantifier rank 3

Ehrenfeucht-Fraïssé Games, I

Given two τ-structures $\mathcal{A}_0$ and $\mathcal{A}_1$ and their powersets $P(A_0)$ and $P(A_1)$.

Two players I (spoiler), II (duplicator)

k numbered pebbles for each structure

Two kind of moves: Set- and point-moves

Play for n moves

i-th move:
I chooses $\alpha \in \{0, 1\}$ and put pebble on an element in $P(A_\alpha)$ (Set-move) or
in A_α (point move).
II puts corresponding pebble on set or point.

Ehrenfeucht-Fraïssé Games, II

After n moves we have from $\mathcal{A}_0$

$$A_1^0, a_2^0, A_3^0, \ldots, a_{n-1}^0, A_n^0$$

and from $\mathcal{A}_1$

$$A_1^1, a_2^1, A_3^1, \ldots, a_{n-1}^1, A_n^1$$

These two sequences are *locally isomorphic* if for all j, k

$$a_k^0 \in A_j^0 \text{ iff } a_k^1 \in A_j^1$$

and for each m-ary $R \in \tau$ and $j_1, j_2, \ldots j_m$

$$R^{\mathcal{A}_0}(a_{j_1}^0, a_{j_2}^0, \ldots, a_{j_m}^0) \text{ iff } R^{\mathcal{A}_1}(a_{j_1}^1, a_{j_2}^1, \ldots, a_{j_m}^1)$$

Lemma: Two sequences in $\mathcal{A}_0$ and $\mathcal{A}_1$ respectively

$$A_1^0, a_2^0, A_3^0, \ldots, a_{n-1}^0, A_n^0$$

and

$$A_1^1, a_2^1, A_3^1, \ldots, a_{n-1}^1, A_n^1$$

are locally isomorphic iff for all quantifierfree formulas B we have

$$\mathcal{A}_0 \models B(A_1^0, a_2^0, A_3^0, \ldots, a_{n-1}^0, A_n^0)$$

iff

$$\mathcal{A}_1 \models B(A_1^1, a_2^1, A_3^1, \ldots, a_{n-1}^1, A_n^1)$$

Proof:
Use induction over the construction of B.

Ehrenfeucht-Fraïssé Games, III

Winning the game:

II wins if the correspondence on the pebbles induces a local isomorphism (including the sets).

Theorem: (Ehrenfeucht-Fraïssé, 1953/61)
II has a winning strategy for the k-pebble n-moves game on $\mathcal{A}_0$ and $\mathcal{A}_1$ iff they satisfy the same $MSOL(\tau)$-sentences with k variables and quantifier depth n.

If no set-moves are played this holds for $FOL(\tau)$.

We write $\mathcal{A}_0 \sim_{k,n}^{MSOL} \mathcal{A}_1$ iff
II has a winning strategy in the game with set moves and
$\mathcal{A}_0 \sim_{k,n}^{FOL} \mathcal{A}_1$ in the game without set moves.

Winning strategies, I

A winning **strategy** is a function which takes a position of length n

$$A_1^0, a_2^0, A_3^0, \ldots, a_{n-1}^0, A_n^0$$

$$A_1^1, a_2^1, A_3^1, \ldots, a_{n-1}^1, A_n^1$$

from $\mathcal{A}_0$ and $\mathcal{A}_1$ respectively
together with a move of player I, say
$X_{n+1}^i \in \{a_{n+1}^i, A_{n+1}^i\}$ as input and returns
$X_{n+1}^{1-i} \in \{a_{n+1}^{1-i}, A_{n+1}^{1-i}\}$ as output such that

$$A_1^0, a_2^0, A_3^0, \ldots, a_{n-1}^0, A_n^0, X_{n+1}^0$$

$$A_1^1, a_2^1, A_3^1, \ldots, a_{n-1}^1, A_n^1, X_{n+1}^1$$

is a winning position
(if it exists, else it is undefined).

Winning strategies, II

Proposition:

$\mathcal{A}_0 \sim_{k,n}^{FOL} \mathcal{A}_1$ and $\mathcal{A}_0 \sim_{k,n}^{MSOL} \mathcal{A}_1$

are *equivalence relations* between τ-structures. I.e., they are **symmetric, reflexive** and **transitive**.

Proof:
Reflexivity: Copy literally
Symmetry: The structures play exchangeable roles (but not the players)
Transitivity: Play on the intermediate board

Winning EF-Games, I

$\tau = \emptyset$

$$\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet$$

$$\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet$$

$\tau = \{R_2\}$, linear orders

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

Winning EF-Games, II

Theorem:

Let $\tau = \emptyset$.
For two sets $\mathcal{A}_0$ and $\mathcal{A}_1$
of size m_0 and m_1 respectively,
we have $\mathcal{A}_0 \sim_{k,n}^{FOL} \mathcal{A}_1$
(in the game without set moves)

iff

$m_0 = m_1$ or
$k \leq m_0$ and $k \leq m_1$

Winning EF-Games, III

Theorem:

Let $\tau = \{R_2\}$.
For two cycle graphs $\mathcal{G}_0$ and $\mathcal{G}_1$
of size v_0 and v_1 respectively,
we have $\mathcal{G}_0 \sim^{FOL}_{k,n} \mathcal{G}_1$
(in the game without set moves)

provided

$v_0 = v_1$ or
$2^k \leq v_0$ and $2^k \leq v_1$

Does the converse hold ?

Ehrenfeucht-Fraïssé Games, IV

Theorem: (Feferman, Vaught, 1956)

If $\mathcal{A}_0 \sim_{k,n}^{MSOL} \mathcal{B}_0$ and $\mathcal{A}_1 \sim_{k,n}^{MSOL} \mathcal{B}_1$
then $\mathcal{A}_0 \sqcup \mathcal{A}_1 \sim_{k,n}^{MSOL} \mathcal{B}_0 \sqcup \mathcal{B}_1$

Theorem: (Feferman, Vaught, 1956)

If $\mathcal{A}_0 \sim_{k,n}^{FOL} \mathcal{B}_0$ and $\mathcal{A}_1 \sim_{k,n}^{FOL} \mathcal{B}_1$
then $\mathcal{A}_0 \times \mathcal{A}_1 \sim_{k,n}^{FOL} \mathcal{B}_0 \times \mathcal{B}_1$

The same holds for "gluing" operations.

Winning EF-Games, IV

Theorem:

Let $\tau = \{R_2\}$.
Let $\mathcal{G}_0$ consist of one cycle of size 2^k
and $\mathcal{G}_1$ consist of two cycles of size 2^k.

Then we have $\mathcal{G}_0 \sim_{k,n}^{FOL} \mathcal{G}_1$
(in the game without set moves)

Corollary:
Connectivity is not FOL-definable in the language of graphs.

Winning EF-Games, V

Now we play the game for $MSOL$ for $\tau = \emptyset$.

2^n elements $2^n - 1$ elements

How many moves does player I need to win?

2^n elements 2^{n-1} elements

How many moves does player I need to win?

Winning EF-Games, VI

The rôle of the pebbles.

How long can we play (without set moves) with **two** pebbles?

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

How long can we play with **three** pebbles?