The complexity of
$\mathfrak{A} \models \phi$

FOL

SOL

$\exists SOL$

$HornSOL$

Computing $\mathcal{M}(\mathfrak{A}, z, \phi)$, I

Given

$\mathfrak{A}$, a τ-structure with $\mid A \mid = m$

$\phi \in FOL(\tau)$ of length n
and quantifier depth q

z an assignment $z : Var_{FOL} \to A$

We want to compute **inductively** the meaning function

$$\mathcal{M}(\mathfrak{A}, z, \phi)$$

and estimate its computational complexity with respect to time and space denoted by

$$TIME(\mathfrak{A}, z, \phi), SPACE(\mathfrak{A}, z, \phi)$$

Computing $\mathcal{M}(\mathfrak{A}, z, \phi)$, II

Recall that τ is purely relational and terms t are either constants or variables.

Atomic formulas:
$R(t_1, t_2, \ldots, t_r)$ with $R \in \tau$ and $t_1 \approx t_2$.

Takes one step in a random access look-up table.
Takes m^r, resp. m^2 steps for searching the table.
One bit space for the result.

Boolean operations:
$\phi = (\phi_1 \wedge \phi_2)$, $\phi = (\phi_1 \vee \phi_2)$, $\phi = \neg\phi_1$

$TIME(\mathfrak{A}, z, \phi) = TIME(\mathfrak{A}, z, \phi_1) + TIME(\mathfrak{A}, z, \phi_2) + 1$
$SPACE(\mathfrak{A}, z, \phi) = max(SPACE(\mathfrak{A}, z, \phi_1), SPACE(\mathfrak{A}, z, \phi_2))$

Computing $\mathcal{M}(\mathfrak{A}, z, \phi)$, III

Quantifiers:
$\phi = \exists x \phi_1(x)$, $\phi = \forall x \phi_1(x)$

We search the structure for an element, hence
$TIME(\mathfrak{A}, z, \phi) = m \cdot TIME(\mathfrak{A}, z, \phi_1)$

We can denote location of search in binary, hence
$SPACE(\mathfrak{A}, z, \phi) = \log m \cdot SPACE(\mathfrak{A}, z, \phi_1)$

Conclusion:

$TIME(\mathfrak{A}, z, \phi) = O(n \cdot m^q)$

$SPACE(\mathfrak{A}, z, \phi) = O(q \cdot \log m)$

Computing $\mathcal{M}(\mathfrak{A}, z, \phi)$, IV

We have considered two problems for FOL:

1. The combined complexity of $\mathcal{M}(\mathfrak{A}, z, \phi)$ where both $\mathfrak{A}$ and ϕ are the input.

   This is in $PSPACE$.

2. The complexity of $\mathcal{M}(\mathfrak{A}, z, \phi)$ for fixed ϕ, where only $\mathfrak{A}$ is the input.

   This is in $\mathbf{P}$ and even in $LOGSPACE \subseteq \mathbf{P}$.

Computing $\mathcal{M}(\mathfrak{A}, z, \phi)$, V

Now we consider SOL.

The only change comes from the quantifiers: Search is over all subsets of A^r.

This takes time 2^{m^r}.

The characteristic function of these sets has size m^r.

Conclusion:

$TIME(\mathfrak{A}, z, \phi) = O(n \cdot 2^{q \cdot m^r})$

$SPACE(\mathfrak{A}, z, \phi) = O(q \cdot \log m^r) = O(q \cdot r \cdot m)$

Computing $\mathcal{M}(\mathfrak{A}, z, \phi)$, VI

We consider two problems for SOL:

1. The combined complexity of $\mathcal{M}(\mathfrak{A}, z, \phi)$ where both $\mathfrak{A}$ and ϕ are the input.

   This is also in $PSPACE$.

2. The complexity of $\mathcal{M}(\mathfrak{A}, z, \phi)$ for fixed ϕ, where only $\mathfrak{A}$ is the input.

   This is in $PSPACE$.

Computing $\mathcal{M}(\mathfrak{A}, z, \phi)$, VII

We want to use now non-deterministic machines.

We denote by $\exists SOL(\tau)$ the set of $SOL(\tau)$-formulas ψ of the form

$$\psi = \exists X_1 \exists X_2 \ldots \exists X_k \phi(X_1, X_2, \ldots, X_k)$$

with $\phi \in FOL(\tau \cup \{X_1, X_2, \ldots, X_k\})$

For fixed $\psi \in \exists SOL$ we have that $\mathcal{M}(\mathfrak{A}, z, \psi)$ is in $\mathbf{NP}$.

Hence, for fixed $\psi \in SOL$ we have that $\mathcal{M}(\mathfrak{A}, z, \psi)$ is in $\mathbf{PH}$,
the **Polynomial Hierarchy**.

The Polynomial Hierarchy, I

We look at **Oracle Turing Machines** OTM. Let X be a problem and C be a class of problems.

We define

$\mathbf{P}^X = \{Y : \exists M$ accepts Y using X as oracle $\}$
$\mathbf{P}^C = \{Y : \exists M$ accepts Y using $X \in C$ as oracle $\}$
Here M is a deterministic polynomial time OTM.

Similarly,

$\mathbf{NP}^X = \{Y : \exists M$ accepts Y using X as oracle $\}$
$\mathbf{NP}^C = \{Y : \exists M$ accepts Y using $X \in C$ as oracle $\}$
Here M is a non-deterministic polynomial time OTM.

The Polynomial Hierarchy, II

We define inductively:

$\Delta_0\mathbf{P} = \Sigma_0\mathbf{P} = \Pi_0\mathbf{P} = \mathbf{P}$

and

$\Delta_{i+1}\mathbf{P} = \mathbf{P}^{\Sigma_i\mathbf{P}}$
$\Sigma_{i+1}\mathbf{P} = \mathbf{NP}^{\Sigma_i\mathbf{P}}$
$\Pi_{i+1}\mathbf{P} = \mathbf{CoNP}^{\Sigma_i\mathbf{P}}$

Finally,

$\mathbf{PH} = \bigcup_{i \in \mathbb{N}} \Sigma_i\mathbf{P}$

Note that $\mathbf{PH} \subseteq PSPACE$ and
$\mathbf{P} = \mathbf{NP}$ iff $\mathbf{P} = \mathbf{PH}$.

Horn formulas, I

A **propositional Horn clause** is a formula of the form

$$\neg p_1 \vee \neg p_2 \vee \ldots \vee \neg p_m \vee q$$

with atmost one variable unnegated.

Equivalently, we can write

$$(p_1 \wedge p_2 \wedge \ldots \wedge p_m \to q)$$

$m = 0$ gives $\mathbf{true} \to q$ and the absence of q gives

$$(p_1 \wedge p_2 \wedge \ldots \wedge p_m \to \mathbf{false})$$

or

$$\neg(p_1 \wedge p_2 \wedge \ldots \wedge p_m)$$

A FOL Horn clause is obtained by replacing variables by atomic formulas.

Horn formulas, II

The **size** $s(C)$ **of a clause** C is the number of variables occurring in Σ.
The **size** $s(\Sigma)$ **of set of clauses** Σ is defined as $\sum_{C \in \Sigma} s(C)$

SAT is the problem of deciding whether a set Σ of clauses with n variables of size m is satisfiable.

Theorem: SAT can be solved in $TIME(2^n \cdot m)$ and is $\mathbf{NP}$-complete.

$HORNSAT$ is like SAT but with Σ a set of Horn clauses.

Theorem: $HORNSAT$ is in $\mathbf{P}$.

Proof: Use unit resolution.

Horn formulas, III

The formulas of $Horn\exists SOL$ are of the form

$$\Psi = \exists X_1 \exists X_2 \ldots \exists X_j \forall x_1 \forall x_2 \ldots \forall x_k \bigwedge_{\ell=1}^{n} \Phi_\ell$$

where each Φ_{ell} is a FOL-Horn clause.

Theorem:(Grädel)
For Ψ a fixed $Horn\exists SOL$ formula $\mathcal{M}(\mathfrak{A}, z, \phi)$ is in $\mathbf{P}$.

We give a proof.

Horn formulas, IV

For simplicity let

$$\Psi = \exists X \forall x_1 \forall x_2 \ldots \forall x_k \bigwedge_{\ell=1}^{n} \Phi_\ell$$

a τ_{graphs}-formula with X r-ary.

So each Φ_ℓ consists of atomic or negated atomic formulas $x_i \approx x_j$, $E(x_i, x_j)$ or $X(x_{i_1}, \ldots, x_{i_r})$.

Let $\mathfrak{A}$ be a structure with elements $a_1, \ldots, a_n$.

There are n_k many assignments for the variables x_i.

Let h be the length of $\Phi = \bigwedge_\ell \Phi_\ell$.

Horn formulas, V

We now form the formula

$$\bigwedge_z subst(\Phi, z)$$

This formula has exactly $h \cdot n^k$ many literals.

In $\mathfrak{A}$ each atomic formula $E(a_i, a_j)$ or$a_i \approx a_j$ is true or false, so we can replace them by $\mathbf{true}$ or $\mathbf{false}$ respectively.

We replace each $X(a_{i_1}, \ldots, a_{i_r})$ by a propositional variable $p_{a_{i_1}, \ldots, a_{i_r}}$.

We obtain so a propositional formula $\bar{\Psi}$.

Horn formulas, VI

Claim 1:
If $\mathfrak{A} \models \Psi$ then $\bar{\Psi}$ is satisfiable.

Proof: Assume $\mathfrak{A} \models \Psi$.
Then there is $U \subset A^r$ such that

$$\mathfrak{A}, U \models \forall x_1 \forall x_2 \ldots \forall x_k \bigwedge_{\ell=1}^{n} \Phi_\ell$$

.

We now define an assignement

$$z(p_{\bar{a}}) = \begin{cases} 1 & \text{if } \bar{a} \in U \\ 0 & \text{if } \bar{a} \notin U \end{cases}$$

Exercise: Show that this z makes $\bar{\Psi}$ true.

Horn formulas, VII

Claim 2:
If $\bar{\Psi}$ is satisfiable then $\mathfrak{A} \models \Psi$.

Proof: Assume z is an assignement which makes $\bar{\Psi}$ true.

We define an interpretation U for X by

$$\bar{a} \in U \text{ iff } z(p_{\bar{a}}) = 1$$

.

Exercise: Show that

$$\mathfrak{A}, U \models \forall x_1 \forall x_2 \ldots \forall x_k \bigwedge_{\ell=1}^{n} \Phi_\ell$$

.

Horn formulas, VII

Proof of Theorem:

- The construction of $\bar{\Psi}$ from Ψ is done in polynomial time
- The size of $\bar{\Psi}$ is polynomial in the size of Ψ.
- Using the polynomial time algorithm for $HORNSAT$, we check the satisfiability of $\bar{\Psi}$.
- Using the lemma, this settles $\mathfrak{A} \models \Psi$.

$2SAT$

Let Σ be a set of propositional clauses of at most two literals each.
These are sometimes called Krom clauses.
Both Horn and Krom are names of Logicians

$2SAT$ is the problem of deciding whether such a Σ is satisfiable.

NL denotes the class of problems decidable in non-deterministic logarithmic space.

Theorem: $2SAT$ is decidable in $\mathbf{NL}$.

$Krom\exists SOL$ is like $Horn\exists SOL$ but with clauses of size 2 rather than Horn cluases.

It is now easy to prove that

Theorem: For fixed $\Psi \in Krom\exists SOL$ the problem $\mathcal{M}(\mathfrak{A}, z\Psi)$ is in $\mathbf{NL}$.
The proof is exactly like for $Horn\exists SOL$.

Definability and Complexity, I

Let K be a class of finite τ-structures.

Let $\mathcal{L}(\tau) \subseteq SOL(\tau)$.
Typically $\mathcal{L}(\tau)$ is one of $Krom\exists SOL(\tau)$, $Horn\exists SOL(\tau)$, $\exists SOL(\tau)$, $SOL(\tau)$, $MSOL(\tau)$,

K is **definable in** $\mathcal{L}(\tau)$ if there exists $\Psi \in \mathcal{L}(\tau)$ such that

$$\mathfrak{A} \in K \text{ iff } \mathfrak{A} \models \Psi$$

Let $\mathbf{C}$ be a complexity class.
typically $LOGSPACE$, $\mathbf{NL}$, $\mathbf{P}$, $\mathbf{NP}$, $\mathbf{PH}$, $PSPACE$

K is in $\mathbf{C}$ iff the problem $\mathfrak{A} \in K$ can be decided with the resources allowed in $\mathbf{C}$.

Definability and Complexity, II

We have shown:

- If K is definable in FOL then $K \in LOGSPACE$.
- If K is definable in $Krom\exists SOL$ then $K \in \mathbf{NL}$.
- If K is definable in $Horn\exists SOL$ then $K \in \mathbf{P}$.
- If K is definable in $\exists SOL$ then $K \in \mathbf{NP}$.
- If K is definable in SOL then $K \in \mathbf{PH}$.

Definability and Complexity, III

We will show in the sequel for **ordered structures**

- (Grädel) If $K \in \mathbf{NL}$, then K is definable in $Krom\exists SOL$.
- (Grädel) If $K \in \mathbf{P}$, then K is definable in $Horn\exists SOL$.

For **arbitrary structures** we have

- (Fagin, Christen) If $K \in \mathbf{NP}$, then K is definable in $\exists SOL$.
- (Meyer and Stockmeyer) If $K \in \mathbf{PH}$, then K is definable in SOL.

$LOGSPACE$

What about FOL-definability and $LOGSPSACE$?

Exercise:
Show that the set of words of even size is not FOL-definable.

Exercise:
Show that the set of words of even size is in $LOGSPACE$.

Exercise:
Conclude that FOL-definability is weaker than decidability in $LOGSPACE$.

Questions:
Which logic corresponds to $LOGSPACE$?
FOL+determinsitic transitive closure

Which complexity class corresponds to FOL?
The circuit complexity class $\mathbf{AC}_0$.