Algorithmic Uses of the Feferman-Vaught Theorem

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by

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Vocabularies, structures and theories

Let τ vocabulary
(or a *similarity type* as Tarski used to call it)
given by a set of relation symbols, but no function symbols nor constants.

$FOL(\tau)$ denotes the set of τ-formulas in First Order Logic.

$SOL(\tau)$ and $MSOL(\tau)$ denote the set of τ-formulas in Second Order and Monadic Second Order Logic.

For a class of τ-structures K
$Th_{FOL}(K)$ is the set of sentences of $FOL(\tau)$ true in all $\mathfrak{A} \in K$.
We write $Th_{FOL}(\mathfrak{A})$ for $K = \{\mathfrak{A}\}$.

Similarly, $Th_{SOL}(K)$ and $Th_{MSOL}(K)$ for SOL and $MSOL$.

A. Tarski and E.W. Beth

Tarski published four short abstracts on model theory in 1949 (Bulletin of the AMS, vol. 55) and had sent his manuscript for

Contribution to the theory of models, I

to E.W. Beth for publication.

Inspired by these, E.W. Beth published two papers on model theory. In one of them he showed that

Theorem:(Beth 1952)
Let $\mathfrak{A}, \mathfrak{B}$ be linear orders,
$\mathfrak{C} = \mathfrak{A} \sqcup_< \mathfrak{B}$ their ordered disjoint union.
Then $Th_{FOL}(\mathfrak{C})$ is uniquely determined by $Th_{FOL}(\mathfrak{A})$ and $Th_{FOL}(\mathfrak{B})$.

In Tarski's school
it was asked in early 1950

Let $\mathfrak{A}, \mathfrak{B}$ be τ-structures,
$\mathfrak{A} \times \mathfrak{B}$ the cartesian product and
$\mathfrak{A} \sqcup \mathfrak{B}$ the disjoint union.

Assume we are given
$Th_{FOL}(\mathfrak{A})$ and $Th_{FOL}(\mathfrak{B})$.

What can we say about
$Th_{FOL}(\mathfrak{A} \times \mathfrak{B})$ and $Th_{FOL}(\mathfrak{A} \sqcup \mathfrak{B})$?

What happens in the case
of infinite sums and products?

This triggered many landmark papers.

It also lead to the study of ultraproducts.

Tarski's pupils dealing with this question

1938 Andrzei Mostowski

A. Mostowski, On direct product theories, JSL 17 (1952), pp. 1-31

1952 Anne Morel

T.E. Frayne, A.C. Morel and D.S. Scott, Reduced direct products, Fundamenta Mathematicae 51 (1962), pp.195-228

1954 Robert Vaught

1957 Solomon Feferman

S. Feferman and R.L. Vaught, The first order properties of algebraic systems, Fundamenta Mathematicae 47 (1959), pp. 57-103

1961 Jerome Keisler

Many papers exploiting ultraproducts

Feferman and Vaught answered

but Tarski did not really appreciate it

Theorem A:(Feferman and Vaught, 1959)
$Th_{FOL}(\mathfrak{A} \times \mathfrak{B})$ and $Th_{FOL}(\mathfrak{A} \sqcup \mathfrak{B})$ are uniquely determined by $Th_{FOL}(\mathfrak{A})$ and $Th_{FOL}(\mathfrak{B})$.

- For $MSOL$ still true for $\mathfrak{A} \sqcup \mathfrak{B}$.
  Ehrenfeucht, Läuchli, Shelah, Gurevich

- Not true for SOL

- By combining it with transductions and interpretations true for a wide variety of **generalized products**.

- Also true for infinite generalized sums and products provided the index structures are sufficiently $MSOL$ indistinguishable.

Ehrenfeucht's proof theorem A

Use Ehrenfeucht-Fraïssé Games

This gives actually more:

Let $q \in \mathbb{N}$ and $FOL^q(\tau)$ denote the sentences of FOL of quantifier rank at most q.

Put $Th^q_{FOL}(\mathfrak{A}) = Th_{FOL}(\mathfrak{A}) \cap FOL^q(\tau)$.

Theorem B:(Feferman and Vaught, 1959)
$Th^q_{FOL}(\mathfrak{A} \times \mathfrak{B})$ and $Th^q_{FOL}(\mathfrak{A} \sqcup \mathfrak{B})$
are uniquely determined by
$Th^q_{FOL}(\mathfrak{A})$ and $Th^q_{FOL}(\mathfrak{B})$.

For $MSOL$ still true for $\mathfrak{A} \sqcup \mathfrak{B}$.

Feferman and Vaught's proof

Use Reduction Sequences

Here one proves by induction, say for disjoint union

Theorem C:(Feferman and Vaught, 1959)
For every formula $\phi \in FOL^q(\tau)$
one can compute in **polynomial time**
a sequence of formulas

$$\langle \psi_1^A, \ldots \psi_m^A, \psi_1^B, \ldots \psi_m^B \rangle \in FOL^q(\tau)^{2m}$$

and there exist a boolean function
$B_\phi : \{0,1\}^{2m} \rightarrow \{0,1\}$ such that

$$\mathfrak{A} \sqcup \mathfrak{B} \models \phi$$

iff

$$B_\phi(b_1^A, \ldots b_m^A, b_1^B, \ldots b_m^B) = 1$$

where $b_j^A = 1$ iff $\mathfrak{A} \models \psi_j^A$ and $b_j^B = 1$ iff $\mathfrak{B} \models \psi_j^B$

Similarly for $MSOL$

Algorithmic problems on finite structures

From 1970 on interest in computing focused on computing on finite structures:

- **NP** complete problems and complexity hierarchies.
- Graph algorithms and other finite data structures.
- Generalizing finite automata to trees and beyond.

Operations on coloured, (un)-directed graphs.

Instead for the disjoint union we can prove theorem B and C for the following operations:

- Concatenation of words, $v \circ w$.
- Joining two trees at a new common root, $T_1 \bullet T_2$.
- H-sums of graphs: For $i = 1, 2$ let $G_i = \langle V(G_i), E(G_i) \rangle$ and $V(G_1) \cap V(G_2) = V(H)$ and $E(H) = E(G_1) \cap V(H)^2 = E(G_2) \cap V(H)^2$.

  Then $G = G_1 \oplus_H G_2$ is given by $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

  and similarly for edge and vertex coloured graphs

$MSOL$-smooth operations

We generalize the previous operations to operations satisfying theorem B or C:

Definition:
A n-ary operation $\mathcal{O}$ on τ-structures is $MSOL$-**smooth** if for every $q \in \mathbb{N}$ and every $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n$

$$Th^q_{MSOL}(\mathcal{O}(\mathfrak{A}_1, \ldots, \mathfrak{A}_n))$$

depends only on $Th^q_{MSOL}(\mathfrak{A}_i)$ for $1 \leq i \leq n$.

$\mathcal{O}$ is **effectively** $MSOL$-**smooth** if there is an algorithm which computes for every $\phi \in MSOL(\tau)$ a **reduction sequence**, i.e. a sequence of formulas as described in theorem C.

Examples of $MSOL$-smooth operations

- The disjoint union is effectively $MSOL$-smooth.

- Quantifier free $MSOL$-**transductions** are effectively $MSOL$-smooth.

- The **fusion operation** $\mathbf{fus}_P$
  i.e. identifying all elements which satisfy
  a given unary predicate P
  is $MSOL$-smooth, but we don't know whether it is effectively so.

Open problem:
Are there more examples?
Are there $MSOL$-smooth operations which are not effectively $MSOL$-smooth ?

$MSOL$-inductiv classes (graph grammars)

Definition:

A class K of τ-structures is $MSOL$-**inductiv** if it is defined inductively using a finite set of $MSOL$-smooth operations.

K is **effectively** $MSOL$-**inductiv** if it is defined inductively using a finite set of effectively $MSOL$-smooth operations.

Open problem:
Are there $MSOL$-inductiv classes K which are not effectively $MSOL$-inductiv ?

Effectively $MSOL$-inductiv classes of structures

- Words $\Sigma^\star$ are defined inductively by
  (i) the empty word is a word
  (ii) one letter words are words
  (iii) words are closed under concatenation

- Coloured trees (forests) are defined similary:
  (i) one leave trees are trees
  (ii) trees are closed under root joining
  (iii) forests are closed under disjoint unions

- Series-parallel (SP) graphs are defined by
  (i) one edge graphs are SP.
  (ii) SP graphs are closed under disjoint unions
  (iii) SP graphs are closed under H-sums for all H with at most two vertices.

More effectively $MSOL$-inductiv classes

- Graphs of **tree width at most** k TW_k can be defined inductively by looking at vertex coloured graphs with at most $k+1$ colours:
  (i) All graphs with at most $k+1$ vertices are in TW_k.
  (ii) TW_k is closed under disjoint union.
  (iii) TW_k is closed under renaming of colours.
  (iv) TW_k is closed under **fusion**, i.e. contraction of all vertices of a specific colour into one vertex.

- Similarly, for graphs of **clique width at most** k CW_k
  (i) All graphs with at most 1 vertex are in CW_k.
  (ii) CW_k is closed under disjoint union.
  (iii) CW_k is closed under renaming of colours.
  (iv) CW_k is closed under adding all possible edges between to sets of differently coloured edes.

Representation of structures in $MSOL$-inductive classes

We can represent the structure $\mathfrak{A}$
by its **relational table**
or by a **parse term** $t_{\mathfrak{A}}$
which displays why it is in K.

In general, finding $t_{\mathfrak{A}}$ is $\mathbf{NP}$-complete.

Theorem E:(Courcelle and M., 2001)
Let K be an $MSOL$-inductiv class of τ-structures
and $\phi \in MSOL(\tau)$.

Given a parse term $t_{\mathfrak{A}}$ for $\mathfrak{A}$, then the problem
of deciding

$$\mathfrak{A} \models \phi$$

can be decided in linear time
(in the size of $t_{\mathfrak{A}}$).

Graph polynomials

A graph polynomial is mapping

$$\mathfrak{p} : \texttt{Graphs} \to \mathcal{R}[X]$$

where $\mathcal{R}$ is a commutative ring and
X is a set of indeterminates.
Very often $\mathcal{R} = \mathbb{Z}$.

So $\mathfrak{p}(G) \in \mathcal{R}[X]$.
We write sometimes $\mathfrak{p}(G; X)$ or $\mathfrak{p}(G; x_1, \ldots, x_m)$
if $X = \{x_1, \ldots x_m\}$.

Furthermore, $\mathfrak{p}$ has to be
invariant under graph isomorphisms.

Examples of graph polynomials

- The chromatic polynomial
- The matching polynomial
- Farrell polynomials
- The Tutte polynomial
- The coloured Tutte polynomial

All these have a fixed
finite set of indeterminates.

Splitting properties of graph polynomials, I

The

- chromatic polynomial,
- matching polynomial,
- Farrell polynomials and
- the (coloured) Tutte polynomials

all satisfy the **multiplicativity property** or **splitting property**:

If $G = G_1 \sqcup G_2$ then

$$\mathfrak{p}(G) = \mathfrak{p}(G_1) \cdot \mathfrak{p}(G_2)$$

Splitting properties of graph polynomials, II

Oxley and Welsh (1992) proved for the Tutte polynomial $T(G)$:

Theorem:
If G_1 and G_2 have one edge e and its two vertices in common, $T(G)$ satisfies for $G = G_1 \sqcup_e G_2$

$$T(G) =$$
$$\alpha_{0,0} \cdot T(G_1 - v) \cdot T(G_2 - v) +$$
$$\alpha_{1,0} \cdot T(G_1 / v) \cdot T(G_2 - v) +$$
$$\alpha_{0,1} \cdot T(G_1 - v) \cdot T(G_2 / v) +$$
$$\alpha_{1,1} \cdot T(G_1 / v) \cdot T(G_2 / v)$$

where the $\alpha_{i,j}$ do not depend on G.

Splitting properties of graph polynomials, III

More generally,
Andrzejak (1997) and Noble (1997) proved:

Theorem:
If G_1 and G_2 have no edges in common, and $V_1 \cap V_2 = X$ has most k vertices, then the $T(G_1 \cup G_2)$ can be computed from the $T(G_1)$ and $T(G_2)$ and Tutte polynomials of graphs derived from G_1, G_2 alone by deletion or contraction of vertices in X.

Their proof does *not generalize easily* to other polynomials.

An analogy

Splitting properties of graph polynomials could be viewed as variants of theorem C with reductions sequences. There

- The ring is boolean.
- The evaluation of the formula ϕ corresponds to the computation of the graph polynomial $\mathfrak{p}$.
- The boolean function B_ϕ corresponds to $g_H^{\mathfrak{p}}(x, y)$
- The reduction sequence corresponds to a sequence of graph polynomials.

Splitting properties of graph polynomials, IV

Let $\mathfrak{p}$ be a graph polynomial,
and H be a fixed graph.

We can ask whether there are

- a finite set of graph polynomials $\mathfrak{P} = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots \mathfrak{p}_m\}$ with $\mathfrak{p} \in \mathfrak{P}$
- and polynomials $g_H^{\mathfrak{p}_i}(x_1, \ldots, x_m, y_1, \ldots, y_m)$

such that for every G, if $G = G_1 \oplus_H G_2$

$$\mathfrak{p}_i(G) = g_H^{\mathfrak{p}_i}(\mathfrak{p}_1(G_1), \ldots \mathfrak{p}_m(G_1), \mathfrak{p}_1(G_2) \ldots \mathfrak{p}_m(G_2),)$$

$\mathfrak{P}$ is called a **splitting set** for $\mathfrak{p}$ and H-sums.

$MSOL$-definable polynomials

After Courcelle, Makowsky, Rotics
and Makowsky, Mariño

We look at polynomials of the form

$$\sum_{\phi(U)} \prod_{(i,j) \in U \subset E} t(i,j)$$

where

- $\phi(U)$ is an $MSOL$-formula,
- and $t(i,j)$ is a weight function with finite range on edges which is obtained by an $MSOL$-definable case distinction.

The precise definition is a bit more general.

The matching polynomial is of this form with

- $\phi(U)$ says that U is a partial matching
- $t(i,j) = x$

The splitting theorem

Theorem F: (M., 2001)
For every graph H, and every $MSOL$-definable graph polynomial $\mathfrak{p}$ there is a (finite) splitting set $\mathfrak{P}$ for $\mathfrak{p}$ and H.

Corollary: (Courcelle, M., Rotics, 2001)
On $MSOL$-inductive classes K
an $MSOL$-definable graph polynomial $\mathfrak{p}$
can be computed in polynomial time
from the parse term t_G of the graph.

More corollaries

Corollary: (M., Mariño 2001)
$MSOL$-Farrell polynomials can be computed in polynomial time on graphs of bounded tree width.

This includes matching polynomials, rook polynomials, but not the Tutte polynomial

Corollary: (M., 2001)
The coloured Tutte polynomial can be computed in polynomial time on graphs of bounded tree width.

improves results by Oxley and Welsh, Andrzejak, Noble, Mighton and gives applications to knot theory

Corollary: (Courcelle, M. 2001)
Theorem E

Remark:
In general these graph polynomials are $\sharp\mathbf{P}$-hard to to compute or even to evaluate.