Definability of graph structures - Tirgul

Operations on graphs

Subgraph: Deleting vertices and edges

Induced subgraph: Deleting vertices (and their adjacent edges)

Minor: Deleting vertices and edges and contracting edges

Topological minor: Deleting vertices and edges and replacing independent paths with edges

Proposition

Let K be a class of graphs which is closed under subgraphs[a] and graph isomorphisms. Then K is characterized by a countable set of forbidden subgraphs.

Proof.
Let F be the set of all graphs $G \notin K$ that are minimal with respect to subgraphs. F is countable since there are only countably many finite graphs.

[a] or induced subgraphs, or minors, or topological minors

Proof cont.

We show that K is exactly the class of all graphs not containing any element in F as a subgraph.

$\Rightarrow$ Assume $G \notin K$. Then there exists a subgraph H of G (the minimal with respect to subgraphs) such that $H \in F$.

$\Leftarrow$ Assume $G \in K$ and H is a subgraph of G, where $H \in F$. Since K is closed under subgraphs we have $H \in K$. A contradiction.

Examples

Bipartite graphs: Forbidden subgraphs: $C_3, C_5, C_7 \ldots$

Forests: Forbidden subgraphs: $C_3, C_4, C_5 \ldots$.
Forbidden minor: C_3.

Chordal graphs: Forbidden induced subgraphs: $C_4, C_5, C_6 \ldots$.

Planar graphs: Forbidden minors: K_5 and $K_{3,3}$ (Kuratowski's theorem).

Theorem

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1. If a class K of graphs is characterized by a *finite* set F of forbidden subgraphs / induced subgraphs then K is FOL-definable.
2. If a class K of graphs is characterized by a *finite* set F of forbidden minors / topological minors then K is $MSOL$-definable.

Proof of Case 1

Let $H = \langle V_H, E_H \rangle \in F$, where $V_H = \{v_1, \ldots v_\ell\}$. We can easily write an FOL-sentence saying that a graph does not contain H as a subgraph:
$\varphi_H = \neg \exists x_1 \exists x_2 \ldots \exists x_\ell \left(\left(\bigwedge_{1 \le i < j \le \ell} x_i \ne x_j \right) \wedge \left(\bigwedge_{(v_i, v_j) \in E_H} E(x_i, x_j) \right) \right)$

or as an induced forbidden subgraph: $\widetilde{\varphi_H} = \neg \exists x_1 \exists x_2 \ldots \exists x_\ell$
$\left(\left(\bigwedge_{1 \le i < j \le \ell} x_i \ne x_j \right) \wedge \left(\bigwedge_{(v_i, v_j) \in E_H} E(x_i, x_j) \right) \wedge \left(\bigwedge_{(v_i, v_j) \notin E_H} \neg E(x_i, x_j) \right) \right)$

As F is finite, K is defined by a finite conjunction of the φ_H's or $\widetilde{\varphi_H}$'s.

Proof of Case 2 (a): Minors

Let $H = \langle V_H, E_H \rangle \in F$, where $V_H = \{v_1, \ldots v_\ell\}$. We show that there exists a formula ϕ_H such that $G \models \phi_H$ iff H is a minor of G.

$$\phi_H = \exists R_1 \cdots \exists R_\ell$$
$$\Big(\big(\bigwedge_i (NotEmp(R_i) \wedge Con(R_i)) \big) \wedge \bigwedge_{i \neq j} Disj(R_i, R_j) \wedge \bigwedge_{(v_i, v_j) \in E} Edge(R_i, R_j) \Big)$$

Every unary relation R_i corresponds to a set of vertices in G contracted to v_i in H. $NotEmp(R_i)$ means that R_i is not empty. $Con(R_i)$ means that R_i is a connected set. $Disj(R_i, R_j)$ means that R_i and R_j are disjoint. $Edge(R_i, R_j)$ means that there exists an edge path between R_i and R_j.

Proof of Case 2 (a): Minors - cont.

For relation variables R, S, define:

$NotEmp(R) = \exists x R(x)$

$Subsetneq(R, S) = \forall x(R(x) \rightarrow S(x)) \wedge \exists x(\neg R(x) \wedge S(x))$
$Disj(R, S) = \neg \exists x(R(x) \wedge S(x))$
$Edge(R, S) = \exists x \exists y(R(x) \wedge S(y) \wedge E(x, y))$

$EClosed(R) = \forall x \forall y((E(x, y) \wedge R(x)) \rightarrow R(y))$

$Con(R) = \neg \exists S(NotEmp(S) \wedge Subsetneq(S, R) \wedge EClosed(S))$

Proof of Case 2 (b): Topological minors

Let $H = \langle V_H, E_H \rangle \in F$, where $V_H = \{v_1, \ldots v_\ell\}$. We show that there exists a formula $\widetilde{\phi_H}$ such that $G \models \widetilde{\phi_H}$ iff H is a topological minor of G.

$$\widetilde{\phi_H} = \exists R_1 \cdots \exists R_\ell \exists P_{i,j}\{(v_i, v_j) \in E\} \left(\left(\bigwedge_i (NotEmp(R_i) \wedge Con(R_i)) \right) \right) \wedge \bigwedge_{i \neq j} Disj(R_i, R_j) \wedge \left(\bigwedge_{(v_i, v_j) \in E} Path(P_{i,j}, R_i, R_j) \right) \wedge \left(\bigwedge_{(v_{i_1}, v_{j_1}), (v_{i_2}, v_{j_2}) \in E} Disj_{i_1, i_2, j_1, j_2}(P_{i_1, j_1}, P_{i_2, j_2}, Ri_1, Ri_2, Rj_1, Rj_2) \right))$$

Every unary relation R_i corresponds to a set of vertices in G contracted to v_i in H. $Path(P_{i,j} R_i, R_j)$ means $P_{i,j}$ contains a path between R_i and R_j, corresponding to a stretching of an edge in H. The formulas $Disj$ determine that the $P_{i,j}$'s intersect only in their endpoints, that is, the R_i's that they connect.

Proof of Case 2 (b): Topological minors - cont.

For relation variables R, S, define:

$Intersect(R, S) = \exists x(R(x) \wedge R(S))$

$Path(R, S) = \exists T(Con(T) \wedge Intersect(R, T) \wedge Intersect(S, T))$

For distinct i_1, i_2, j_1, j_2 we have:
$Disj_{i_1,i_2,j_1,j_2}(P_{i_1,j_1}, P_{i_2,j_2}, R_{i_1}, R_{i_2}, R_{j_1}, R_{j_2}) = \neg\exists x(P_{i_1,j_1}(x) \wedge P_{i_2,j_2}(x))$

For distinct j_1, j_2 we have:
$Disj_{i,i,j_1,j_2}(P_{i,j_1}, P_{i,j_2}, R_i, R_i, R_{j_1}, R_{j_2}) = \neg\forall x((P_{i,j_1}(x) \wedge P_{i,j_2}(x)) \rightarrow R_i(X))$

Etc.

The Graph Minor Theorem
(Robertson and Seymour)

Every minor closed class of graphs is defined by a FINITE set of forbidden minors.