The Monadic Quantifier Alternation Hierarchy over Grids and Pictures

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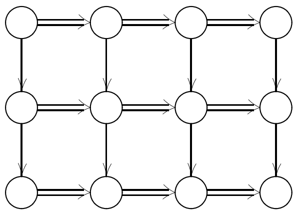
Abstract. The subject of this paper is the expressive power of monadic second-order logic over two-dimensional grids. We give a new, self-contained game-theoretical proof of the nonexpressibility results of Matz and Thomas. As we show, this implies the strictness of the monadic second-order quantifier alternation hierarchy over grids.

1 Introduction

Grids are finite structures with two edge relations whose vertices are arranged as the elements of a matrix. The two edge relations are successor relations connecting each vertex with the corresponding vertex in the following row, respectively column. We can visualize a grid in the following intuitive way:



Pictures can be viewed as "two-dimensional words". Formally, they are coloured grids in which each vertex is labelled with exactly one out of finitely many colours; i.e. pictures are grids that are equipped with some additional unary relations.

Monadic second-order logic (MSO, for short) is the fragment of second-order logic in which second-order quantifiers may only range over *monadic* predicates, i.e. over sets.

Viewed over finite structures, there is a close relationship between (generalized) *automata theory* and certain fragments of monadic second-order logic:

Büchi and Elgot ([Bue60,Elg61]) proved that a *word*-language is recognizable by a finite automaton if and only if it can be characterized by a MSO-formula (or, equivalently, even by an existential MSO-formula). Thatcher and Wright ([TW68]) and Doner ([Don70]) were able to generalize this result to *trees*. Giammarresi, Restivo, Seibert and Thomas ([GRST96]) proved that a *picture*-language is recognizable by a tiling-system (which can be viewed as a finite automaton over pictures) if and only if it can be characterized by an existential MSO-formula.

Concerning desriptive *complexity theory*, from Fagin and Stockmeyer ([Fag74], [Sto77]) we know that for each $k \geqslant 1$, the kth level Σ_k^P of the polynomial time hierarchy coincides with the class of all sets of finite structures that can be defined by a second-order formula which has a prefix of k alternating blocks of second-order quantifiers, starting with an existential block, and followed by a first-order formula (Σ_k^1-formula, for short). This, at least in principle, makes it possible to prove the strictness (or the collapse) of the polynomial time hierarchy by studying the expressive power of Σ_k^1-formulas (for $k \geqslant 1$).

Ajtai, Fagin and Stockmeyer ([AFS97]) and, independently, Makowsky and Pnueli ([MP94]) obtained the result that for each $k \geqslant 1$ there is a problem which is both, complete for the kth level of the polynomial time hierarchy and definable by a *monadic* Σ_k^1-formula.

Thus, under the complexity-theoretic assumption that the polynomial time hierarchy is strict, we obtain the strictness of the *monadic hierarchy*, where the kth level of the monadic hierarchy consists of all sets of finite structures that can be defined by a *monadic* Σ_k^1-formula. (In the following, we will shortly write Σ_k to denote the class of all *monadic* Σ_k^1-formulas.)
In [Fag94] Fagin raised the question whether one can prove the strictness of the monadic hierarchy without invoking such a complexity-theoretic assumption.

Let us collect some results concerning Fagin's problem:

As already mentioned, Büchi and Elgot (resp. Thatcher, Wright and Doner) obtained that over words (and trees) full monadic second-order logic has the same expressive power as its existential fragment. I.e. over words and trees the monadic hierarchy collapses to its first level. Additionally, in [Th82] Thomas proved that even one single existential quantifier suffices.

In [Ott95] Otto showed that over certain grid-like structures within the first level of the monadic hierarchy, the number of existential set quantifiers matters, i.e. within Σ_1 more set quantifiers lead to more expressive power.

In [Fag75] (see also [FSV95]), Fagin proved that graph connectivity cannot be expressed in Σ_1. As this problem can easily be expressed by a Π_1-formula, he thus separated the first two levels of the monadic hierarchy over graphs.

In [GRST96] Giammarresi, Restivo, Seibert and Thomas proved that the class of Σ_1-definable picture-languages is not closed under complementation, and thus separated the first two levels of the monadic hierarchy over pictures.

Finally, in [MT97] Matz and Thomas fully answered Fagin's question: They proved that the monadic hierarchy is strict over graphs and infinite over grids. In their paper, they asked whether one can prove the *strictness* of the monadic hierarchy over grids, as well.[1]

[1] Note that over (uncoloured) grids, the infinity of the monadic hierarchy does not trivially imply the strictness.

By a refinement of their methods, the strictness of the monadic hierarchy over grids was established in [Schw97].

In order to prove the infinity of the monadic hierarchy, Matz and Thomas showed for every $k \geqslant 1$ that some MSO-definable set of grids cannot be defined by a Σ_k-sentence. Their proof is based on the main theorem of [GRST96] and some automata theoretic machinery, which was a rather new and unconventional method for proving nonexpressibility results.
The main contribution of the present paper is an alternative nonexpressibility proof by means of a modified Ehrenfeucht-Fraïssé game. Apart from being self-contained, it also has the advantage of generalizing to $\leqslant_1$-S_2-grids, i.e. to grids in which one of the successor relations is replaced by a linear ordering.

In detail, the present paper is structured as follows:

In section 2 we will provide the notation necessary to formulate the main results of [MT97] and [Schw97], presented in section 3. A joint paper containing detailed proofs of these results is in preparation.

To prove the infinity of the monadic hierarchy over grids, Matz and Thomas investigated sets of grids in which for each height the set contains exactly one grid of that height, i.e. sets of grids where the width of the grids is a function of their height. Using concepts of automata theory and applying the main theorem of [GRST96], they showed that functions which are Σ_k-definable (in the sense that the corresponding sets of grids are Σ_k-definable), are at most k-fold exponential. In section 5 we will present a more streamlined approach, namely we will design and analyse an appropriate Ehrenfeucht-Fraïssé game replacing their automata theoretical argument to reprove their upper bound on the growth rate of Σ_k-definable functions.

Together with the result (see Proposition 1 below) that some function growing too fast for Σ_k can be defined in Σ_{k+1}, one thus obtains the strictness of the monadic hierarchy over grids. In section 4 we will briefly explain the idea of how to obtain that result. Finally, in section 6 we will discuss further results concerning some extended versions of grids.

Let us mention that the problems witnessing the strictness of the monadic hierarchy over grids all lie in the complexity class P and, in particular, in the first level of the polynomial time hierarchy. On the other hand Matz, Thomas, and the author were able to show (see Theorem 1 below) that monadic second-order logic over coloured grids allows to define complete problems for each level of the polynomial time hierarchy.

I want to thank Clemens Lautemann for the help and advice he gave me during the preparation of both this paper and my diploma thesis, Thomas Schwentick for helpful remarks on the Ehrenfeucht-Fraïssé game presented in section 5, and Wolfgang Thomas and Oliver Matz for inspiring discussions on the monadic hierarchy.

2 Notation

$\mathbb{N} := \{1, 2, \ldots\}$ is the set of all positive integers. Indices m, n, i, j etc. always refer to numbers in $\mathbb{N}$. With $[m,n] := \{1, \ldots, m\} \times \{1, \ldots, n\}$ we denote the set of all tuples (i,j) for $1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n$.

2.1 Grids and Pictures

Definition 1 (Grid). *Let $m, n \geqslant 1$. The* grid $\underline{[m,n]}$ *of height m and width n is the structure* $\underline{[m,n]} := ([m,n], S_1^{m,n}, S_2^{m,n})$, *where*
$S_1^{m,n} := \{((i,j),(i+1,j)) : 1 \leqslant i < m,\ 1 \leqslant j \leqslant n\}$ *is the* row-successor *relation (which connects each element to the proper element in the following row), and*
$S_2^{m,n} := \{((i,j),(i,j+1)) : 1 \leqslant i \leqslant m,\ 1 \leqslant j < n\}$ *is the* column-successor *relation (which connects each element to the proper element in the following column).*

The elements in $[m,n]$ are called *vertices.*
If we symbolize vertices by circles $\bigcirc$, edges in S_1 by arrows $\downarrow$, and edges in S_2 by arrows $\Longrightarrow$, we can represent the grid $\underline{[3,4]}$ as visualized in the introduction.
The expressions *row, column, top, bottom* etc. are interpreted in the usual way, e.g. the leftmost column contains exactly the vertices $(i,1)$ for $1 \leqslant i \leqslant m$, and the top row consists of all vertices $(1,j)$ for $1 \leqslant j \leqslant n$.

We view a grid which additionally has t unary relations $X_1, .., X_t$ as a coloured grid (i.e. a *picture*) in which each vertex is labelled by exactly one out of 2^t possible colours.

Definition 2 (t-bit picture). *Let $t \geqslant 0$. A t*-bit picture $\underline{P}$ *is a structure* $\underline{P} := (\underline{[m,n]}, X_1^P, .., X_t^P)$, *where* $\underline{[m,n]}$ *is a grid of height m and width n, and $X_i^P \subseteq [m,n]$ is a set of vertices for every $i \leqslant t$.*
We define t-pic *to be the class of all t-bit pictures.*

The colour of a vertex $v \in [m,n]$ is given by the sets $X_1^P, .., X_t^P$ as follows: For each $i \leqslant t$ let $c_i(v) := 1$ in case $v \in X_i^P$, and $c_i(v) := 0$ in case $v \notin X_i^P$. Then v is coloured by the string $c_1(v) \cdots c_t(v) \in \{0,1\}^t$, i.e. the ith set X_i^P determines the ith bit $c_i(v)$ of the colour-string $c_1(v) \cdots c_t(v)$ of each vertex v. Note that (uncoloured) grids are 0-bit pictures.

2.2 Monadic Second-Order Logic over Grids

Monadic second-order formulas over grids are built of two binary relation symbols S_1, S_2 (for row- and column-successor), the equality symbol $=$, the logical connectives $\neg, \vee$, individual variables (i.e. first-order variables) $x, y, x_1, x_2, \ldots$, set variables (i.e. monadic second-order variables) $X, Y, X_1, X_2, \ldots$, and the existential quantifier $\exists$ which may quantify both individual and set variables.
As done e.g. in [EF95], we will use $\forall x \varphi$ (resp. $\forall X \varphi$, $\varphi \wedge \psi$, $\varphi \rightarrow \psi$, $\varphi \leftrightarrow \psi$)

as abbreviation for $\neg\exists x \neg\varphi$ (resp. $\neg\exists X \neg\varphi$, $\neg(\neg\varphi \vee \neg\psi)$, $\neg\varphi \vee \psi$, $(\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$).

By MSO we denote the class of all monadic second-order formulas over grids. FO-formulas are MSO-formulas in which no set-quantifier occurs. With $\mathrm{free}(\varphi)$ we denote the set of all individual and set variables occuring free (i.e. not in the scope of some quantifier) in $\varphi \in \mathrm{MSO}$.
Let $\varphi \in \mathrm{MSO}$ with $\mathrm{free}(\varphi) = \{X_1, .., X_t, x_1, .., x_s\}$, $\underline{P} = (\underline{[m,n]}, X_1^P, .., X_t^P)$ a t-bit picture and $x_1^P, .., x_s^P$ vertices in $\underline{P}$. We write $(\underline{P}, x_1^P, .., x_s^P) \models \varphi$ iff φ is satisfied when interpreting the free occurrences of the variables $X_1, .., X_t, x_1, .., x_s$ with the values $X_1^P, .., X_t^P, x_1^P, .., x_s^P$, and interpreting S_1 and S_2 as the row- and the column-successor relation of the grid, respectively. (See e.g. [EF95] for a precise definition of the syntax and semantics of monadic second-order logic.)

Example 1. $\mathit{left}(x) := \neg\exists y S_2 yx$ is a FO-formula asserting that vertex x is in the leftmost column of a grid. Of course, there are similar FO-formulas $\mathit{right}(x)$, $\mathit{top}(x)$, $\mathit{bottom}(x)$, $\mathit{topleft}(x)$, etc. asserting that vertex x is in the rightmost column, top row, bottom row, in the top left position, respectively.

Definition 3. *Let* $t \geqslant 0$, $\varphi \in \mathrm{MSO}$. *If* $\mathrm{free}(\varphi) \subseteq \{X_1, .., X_t\}$, *then with* $\mathrm{MOD}_{t\text{-pic}}(\varphi) := \{\underline{P} : \underline{P} \models \varphi$ *and* $\underline{P}$ *is a* t*-bit picture*$\}$ *we denote the class of all* t*-bit pictures satisfying* φ.
If $\mathrm{free}(\varphi) \nsubseteq \{X_1, \ldots, X_t\}$, *then we define* $\mathrm{MOD}_{t\text{-pic}}(\varphi) := \emptyset$.
In case $t = 0$ *we write* $\mathrm{MOD}_{\mathrm{grids}}(\varphi) := \mathrm{MOD}_{0\text{-pic}}(\varphi)$ *to denote the class of all (uncoloured) grids satisfying* φ.

A class $\mathcal{C}$ of t-bit pictures is called MSO-definable iff there is a MSO-formula φ such that $\mathcal{C} = \mathrm{MOD}_{t\text{-pic}}(\varphi)$.

Definition 4.

1. *We define the following fragments of* MSO *for every* $k \geqslant 1$*:*
   $\Sigma_0 := \mathrm{FO}$,
   $\Sigma_k := \{\exists X_1, .., X_r \neg\varphi : \varphi \in \Sigma_{k-1},\ r \geqslant 0,$ *and* $X_1, .., X_r$ *are set variables*$\}$,
   $\Pi_k := \{\neg\varphi : \varphi \in \Sigma_k\}$.
   Thus Π_k *is the set of all negated* Σ_k*-formulas, and* Σ_k *consists of all* MSO*-formulas having a prefix of* k *alternating blocks of set quantifiers, starting with an existential block, and followed by a first-order formula.*
2. *For a set* Γ *of* MSO*-formulas we define*
   $$\underline{\Gamma}[\mathrm{grids}] := \{\mathrm{MOD}_{\mathrm{grids}}(\varphi) : \varphi \in \Gamma\},$$
   $$\underline{\Gamma}[t\text{-pic}] := \{\mathrm{MOD}_{t\text{-pic}}(\varphi) : \varphi \in \Gamma\}$$
   to be the class of all sets of grids (resp. t*-bit pictures) definable in* Γ.
   The class $\underline{\Sigma_k}[\mathrm{grids}]$ *(resp.* $\underline{\Sigma_k}[t\text{-pic}]$*) is called the* k*th level of the monadic hierarchy over grids (resp.* t*-bit pictures).*

Example 2. For $X \subseteq [m,n]$ and $v \in [m,n]$ let $c_X(v) := 1$ in case $v \in X$, and $c_X(v) := 0$ in case $v \notin X$. For each column $j \in \{1,..,n\}$ of the grid $\underline{[m,n]}$ we can interpret the string $c_X(j) := c_X(1,j) \cdots c_X(m,j) \in \{0,1\}^m$ as the binary representation of an integer $\boldsymbol{c}_X(j) \in \{0,1,..,2^m-1\}$ which can be viewed as the number in column j, induced by X.

The FO-formula $col\text{-}num(X) :=$

$$\begin{array}{ll} \forall x_0, y_0 & (\{\, bottom(x_0) \wedge S_2 x_0 y_0 \} \rightarrow \{\, X x_0 \leftrightarrow \neg X y_0 \,\}) \wedge \\ \forall x_0, y_0, x_1, y_1 & (\{\, S_1 x_1 x_0 \wedge S_1 y_1 y_0 \wedge S_2 x_0 y_0 \,\} \rightarrow \\ & \quad \{\, (\, (X x_0 \wedge \neg X y_0) \rightarrow (X x_1 \leftrightarrow \neg X y_1)) \wedge \\ & \quad \;\; (\, (\neg (X x_0 \wedge \neg X y_0)) \rightarrow (X x_1 \leftrightarrow X y_1)) \quad \}) \end{array}$$

asserts that every two successive columns of a grid $\underline{[m,n]}$ have (modulo 2^m) successive column-numbers, i.e. $\boldsymbol{c}_X(j+1) \equiv \boldsymbol{c}_X(j) + 1 \pmod{2^m}$ for all $j < n$.

Hence for each $m \geqslant 1$ the Σ_1-formula

$$\begin{array}{lll} exp & := \exists X \; \{ \; col\text{-}num(X) \wedge & \\ & \qquad \forall x \;\; ((left(x) \rightarrow \neg X x) \;\wedge\; (right(x) \rightarrow X x)) \wedge & \\ & \qquad \forall x, y \;\; ((top(x) \wedge S_2 x y) \rightarrow (X y \vee \neg X x)) & \} \end{array}$$

is satisfied exactly by the grid of width 2^m; i.e.
$\mathrm{MOD}_{\mathrm{grids}}(exp) = \{\underline{[m, 2^m]} \; : \; m \geqslant 1\}$.

2.3 Definability and Growth Rate of Functions

A function $f : \mathbb{N} \longrightarrow \mathbb{N}$ is called Σ_k-definable iff there is a sentence $\varphi \in \Sigma_k$ such that $\mathrm{MOD}_{\mathrm{grids}}(\varphi) = \{\, \underline{[m, f(m)]} \; : \; m \geqslant 1 \,\}$, i.e. for each $m \geqslant 1$ there is exactly one grid of height m, namely the grid of width $f(m)$, which satisfies φ.

Definition 5. *Let* $s_0(m) := m$, $s_k(m) := 2^{s_{k-1}(m)}$ *for every* $k, m \geqslant 1$.
A function $f : \mathbb{N} \longrightarrow \mathbb{N}$ *is called*

- at most k-fold exponential *if* $f(m) \leqslant s_k(\mathcal{O}(m))$, *i.e. there is a constant* $c > 0$ *such that* $f(m) \leqslant s_k(cm)$ *for all* $m \geqslant 1$.
- k-fold exponential *if* $f(m) \leqslant s_k(\mathcal{O}(m))$ *and* $f(m) \nleqslant s_{k-1}(\mathcal{O}(m))$.

In particular, from Example 2 we know that the function s_1 (where $s_1(m) = 2^m$) is Σ_1-definable.

3 Main Results

We can now state the results of [MT97] and [Schw97] formally.
With Σ_k^P we denote the kth level of the polynomial time hierarchy.
Matz, Thomas, and the author obtained the following result:

Theorem 1. *For each* $k \geqslant 1$ *there exists a set of* 3*-bit pictures which is both complete for the* k*th level of the polynomial time hierarchy (with respect to polynomial time reductions) and definable by a* Σ_k*-formula, i.e.* $\underline{\Sigma_k}$ [3-pic] *contains a* Σ_k^P*-complete problem.*

Proof idea. It is well-known (cf. [Sto77]) that validity of quantified Boolean formulas with k blocks of quantifiers is complete for the kth level of the polynomial time hierarchy. Our basic idea is to represent such a quantified Boolean formula φ in conjunctive normal form by a coloured grid which has a row for each variable and a column for each clause, such that the vertex in row i and column j has colour P (resp. N) if variable v_i occurs unnegated (resp. negated) in the jth clause of the formula. The k different blocks of variables can be indicated by an additional colour B. Then, an assignment to the variables in φ corresponds to a colouring of the corresponding grid, where vertices in the same row have the same colour; and validity of φ can be expressed by a Σ_k-formula. $\square$

In particular, monadic second-order logic over 3-bit pictures allows to define complete problems for each level of the polynomial time hierarchy.

Our main result is the strictness of the monadic hierarchy over grids and over t-bit pictures. In order to prove it, we follow the strategy of Matz and Thomas in [MT97]. In section 5 we will reprove their following theorem, replacing their automata-theoretic proof by an argument based on an Ehrenfeucht-Fraïssé game.

Theorem 2 (Matz, Thomas).
Σ_k-definable functions are at most k-fold exponential.

Matz and Thomas then showed that some k-fold exponential function is Σ_{2k+3}-definable, concluding that the monadic hierarchy over grids is infinite. We can improve this to

Proposition 1. *Let $f_1(m) = 2^m$, $f_{k+1}(m) = f_k(m)2^{f_k(m)}$ for every $k, m \geqslant 1$. For all $k \geqslant 1$, the k-fold exponential function f_k is definable in Σ_k and in Π_k.*

A short proof idea of Proposition 1 will be given in section 4. From Theorem 2 and Proposition 1 we conclude that $\underline{\Sigma_k}\,[\text{grids}] \not\ni \{\underline{[m, f_{k+1}(m)]} : m \geqslant 1\} \in \underline{\Sigma_{k+1}}\,[\text{grids}]$ and thus directly obtain

Theorem 3.
$\underline{\Sigma_k}\,[\text{grids}] \subsetneqq \underline{\Sigma_{k+1}}\,[\text{grids}]$ *and* $\underline{\Sigma_k}\,[t\text{-pic}] \subsetneqq \underline{\Sigma_{k+1}}\,[t\text{-pic}]$ *for all* $k, t \geqslant 1$.
I.e., the monadic second-order quantifier alternation hierarchy over grids and t-bit pictures is strict.

4 Σ_k-definability of a fast-growing Function

In this section we briefly explain the idea of how to prove Proposition 1.
A detailed exposition can be found in [Schw97].

In Example 2 we have seen that the function $f_1(m) := 2^m$ is Σ_1-definable.
For $k > 1$ one can use a similar idea. Our Σ_k-formula defining the function $f_k(m) := f_{k-1}(m)2^{f_{k-1}(m)}$ describes the following counting process:

Let a grid of height m and width n be given.
<u>**Step 1**</u>: As done in Example 2, starting with 0^m we write successive binary numbers of length m into the columns of the grid. If n is a multiple of 2^m then the rightmost column has number 1^m (and the numbering is called a *complete column-numbering*). In this case the grid can be divided into blocks of width $2^m = f_1(m)$, and the columns with number 0^m mark the starting points of these blocks.
<u>**Step i+1**</u> ($1 \leqslant i < k$): In the previous step the grid has been divided into blocks of width $f_i(m)$. Starting with $0^{f_i(m)}$ we write successive binary numbers of length $f_i(m)$ into the top row of the grid. If n is a multiple of $f_i(m)2^{f_i(m)} = f_{i+1}(m)$ then the rigthmost block has number $1^{f_i(m)}$ (and the numbering is called a *complete f_i-numbering*). In this case the grid can be divided into blocks of width $f_{i+1}(m)$, and the starting points of the subblocks with number $0^{f_i(m)}$ mark the starting points of these larger blocks.

Our formula defining the function f_k states that there exist sets $C, N_1, .., N_{k-1}$, $M_1, .., M_k$ such that C is a complete column-numbering, M_1 consists of the vertices marking the starting points of the blocks of width $f_1(m)$, for all $i < k$, N_i is a complete f_i-numbering and M_{i+1} consists of the vertices marking the starting points of the blocks of width $f_{i+1}(m)$, and M_k consists of exactly one element.

From Example 2 we obtain a FO-formula asserting that C is a complete column-numbering. The crucial point is to obtain formulas φ_i asserting that N_i is a complete f_i-numbering. For our formula φ_i we assume that N_{i-1} already is a complete f_{i-1}-numbering and that M_i consists of the starting points of the blocks of width $f_i(m)$. Of course, N_i is a complete f_i-numbering if and only if the leftmost (resp. rightmost) block of width $f_i(m)$ entirely consists of zeroes (resp. ones) and every two successive blocks of width $f_i(m)$ have successive N_i-numbers.

We compare the N_i-numbers of two successive blocks of width $f_i(m)$ by comparing all corresponding subblocks of width $f_{i-1}(m)$. As N_{i-1} is a complete f_{i-1}-numbering we can use the N_{i-1}-number of a subblock as its address within the larger block.

The following is a simple formalization of what we usually do when adding 1 to a q-ary number; where, in our case, q equals $2^{f_{i-1}(m)}$ and the separate bits of a q-ary number correspond to the subblocks of width $f_{i-1}(m)$.

The N_i-number of a block Y of width $f_i(m)$ is (modulo $2^{f_i(m)}$) one larger than the N_i-number of a block X of width $f_i(m)$ if and only if the following holds:

1. The N_i-number of the rightmost subblock in Y is (modulo $2^{f_{i-1}(m)}$) one larger than the N_i-number of the rightmost subblock in X, and
2. for every two successive subblocks Y_1 and Y_0 in Y (resp. X_1 and X_0 in X) of width $f_{i-1}(m)$ such that Y_0 and X_0 have the same N_{i-1}-address, the following holds: If the N_i-number of Y_0 consists of zeroes (i.e., is equal to 0) and the N_i-number of X_0 consists of ones (i.e., is equal to $2^{f_{i-1}(m)} - 1$), *then*

the N_i-number of Y_1 is (modulo $2^{f_{i-1}(m)}$) one larger than the N_i-number of X_1 ; *else*: Y_1 and X_1 have equal N_i-numbers.

A formalization of this idea inductively leads to a Π_i-formula asserting that two blocks of width $f_i(m)$ have successive N_i-numbers.
A closer look at the technical details of the proof shows that, applying the method described above, one can construct a Σ_k-formula (and a Π_k-formula) defining the k-fold exponential function f_k.

5 An Ehrenfeucht-Fraïssé Game on Pictures – Nonexpressibility Results for $\mathbf{\Sigma_k}$

Ehrenfeucht-Fraïssé games ([Ehr61,Fra54]) are played on two structures (e.g. pictures) by two players, called Spoiler and Duplicator. Spoiler intends to show a difference between both structures while Duplicator tries to let them look alike. An introduction to Ehrenfeucht-Fraïssé games can be found in [EF95]. In order to prove Theorem 2, we introduce the following kind of game. Let $k, q, r, t \geqslant 0$.

The (k, q, r)-game on two t-bit pictures $\underline{A}$ and $\underline{B}$
k is the number of "second-order rounds", q is the number of "first-order rounds", and r is the number of sets available in each second-order round to colour both pictures. The (k, q, r)-game is divided into two parts:
Part 1 k second-order rounds:
In each round Spoiler chooses one of the two pictures and colours it by choosing r sets of vertices in this picture. Duplicator responds by choosing r sets of vertices in the other picture.
Part 2 q first-order rounds:
In each round Spoiler chooses a vertex in one of the two pictures. Duplicator responds by choosing a vertex in the other picture.

At the beginning the t-bit pictures $\underline{A}$ and $\underline{B}$ are already coloured by sets $X_1^A, \ldots, X_t^A$ and $X_1^B, \ldots, X_t^B$. Let $X_{t+1}^A, \ldots, X_{t+kr}^A$ be the sets chosen to colour $\underline{A}$ during the second-order rounds, and $X_{t+1}^B, \ldots, X_{t+kr}^B$ the sets chosen to colour $\underline{B}$, indexed with respect to the order in which they were chosen.
For each $j \in \{1, \ldots, q\}$ let x_j^A be the vertex chosen in $\underline{A}$ and x_j^B the vertex chosen in $\underline{B}$ in the jth first-order round.

Duplicator wins iff the following three winning conditions hold:

1. Considering only the vertices $x_1^A, \ldots, x_q^A$ and $x_1^B, \ldots, x_q^B$, the pictures $\underline{A}$ and $\underline{B}$, extended by the new colours chosen in the second-order rounds, look the same. More precisely:
   - $x_j^A \in X_i^A \iff x_j^B \in X_i^B$ for every $j \leqslant q,\ i \leqslant t + kr$.
   - $(x_j^A, x_{j'}^A) \in S_i^A \iff (x_j^B, x_{j'}^B) \in S_i^B$ for every $j, j' \leqslant q,\ i \in \{1, 2\}$, i.e. within the grid-structure $x_{j'}^A$ is the row- (respectively column-) successor of x_j^A iff $x_{j'}^B$ is the row- (respectively column-) successor of x_j^B.

- $x_j^A = x_{j'}^A \iff x_j^B = x_{j'}^B$ for all $j, j' \leqslant q$.

2. On the last 2^q columns Duplicator played *column-consistent*, i.e. in each first-order round Duplicator chose a vertex within the last 2^q columns if and only if Spoiler had chosen a vertex within the last 2^q columns; and whenever Spoiler had chosen a vertex within the last 2^q columns, then Duplicator chose a vertex having the same distance from the rightmost column.
   More precisely: If $\underline{A}$ is of width n^A and $\underline{B}$ of width n^B and if x_j^A is in column $(n^A - \beta_j^A)$ and x_j^B is in column $(n^B - \beta_j^B)$, then for all $j \leqslant q$ we have
   - $\beta_j^A < 2^q \iff \beta_j^B < 2^q$ and
   - if $\beta_j^A < 2^q$ then $\beta_j^A = \beta_j^B$.
3. The last 2^q columns of $\underline{A}$ and $\underline{B}$, extended by the new colours chosen during the second-order rounds, are coloured identically. *In particular, $\underline{A}$ and $\underline{B}$ must have the same height.*
   More precisely: If $\underline{A}$ is of height m^A and width n^A and $\underline{B}$ of height m^B and width n^B, then
   - $m^A = m^B =: m$ and
   - for every $i \leqslant t + kr$ and for each row $\alpha \in \{1, .., m\}$ and each distance $\beta \in \{0, .., 2^q - 1\}$ from the rightmost column of the grid, in $\underline{A}$ the vertex in row α and column $(n^A - \beta)$ belongs to X_i^A if and only if in $\underline{B}$ the vertex in row α and column $(n^B - \beta)$ belongs to X_i^B.

We write $\underline{A} \underset{k,q,r}{\equiv} \underline{B}$ iff Duplicator has a winning strategy in the (k, q, r)-game on $\underline{A}$ and $\underline{B}$, i.e. if, no matter which sets and vertices Spoiler chooses, Duplicator can always respond in such a way that she wins the game.
Note that $\underset{k,q,r}{\equiv}$ is an equivalence relation on the class of all pictures.

Definition 6 (Equivalence Class).
For a t-bit picture $\underline{A} \in t\text{-pic}$, by $\langle \underline{A} \rangle_k^{q,r,t} := \{ \underline{B} \in t\text{-pic} : \underline{A} \underset{k,q,r}{\equiv} \underline{B} \}$ we denote the equivalence class of $\underline{A}$, i.e. the class of all t-bit pictures $\underline{B}$ such that Duplicator has a winning strategy in the (k, q, r)-game on $\underline{A}$, $\underline{B}$.
With $e_k^{q,r,t}(m) := |\{ \langle \underline{A} \rangle_k^{q,r,t} : \underline{A}$ is a t-bit picture of height $m \}|$ we denote the number of equivalence classes in the (k, q, r)-game on t-bit pictures of height m.[2]

For a first-order formula φ, let $\mathrm{qd}(\varphi)$ denote the quantifier depth of φ.

Definition 7. *Let $k \geqslant 1$, $q, r \geqslant 0$.*

1. $\mathrm{FO}^q := \{ \varphi \in \mathrm{FO} : \mathrm{qd}(\varphi) \leqslant q \}$
   denotes the set of all first-order formulas of quantifier depth at most q.
2. $\Sigma_0^{q,r} := \mathrm{FO}^q$
   $\Sigma_k^{q,r} := \{ \exists X_1, .., X_{r'} \neg\varphi : \varphi \in \Sigma_{k-1}^{q,r},\ r' \leqslant r,$ *and* $X_1, .., X_{r'}$ *are set variables* $\}$
   $\Pi_k^{q,r} := \{ \neg\varphi : \varphi \in \Sigma_k^{q,r} \}$;
   i.e. $\Sigma_k^{q,r}$ is the set of all Σ_k-formulas φ such that each block of second-order

[2] Here, for an infinite set I we define $|I| := \infty$.

quantifiers consists of at most r set variables, and the first-order part of φ has quantifier depth at most q. $\Pi_k^{q,r}$ is the set of all negated $\Sigma_k^{q,r}$-formulas.

3. *Let Γ be a set of* MSO*-formulas (e.g. $\Gamma = \Sigma_k^{q,r}$).*
   With $\mathrm{B}(\Gamma)$ we denote the set of all boolean combinations of formulas in Γ, i.e. $\mathrm{B}(\Gamma)$ is the smallest set of formulas which satisfies the following conditions: (a) $\Gamma \subseteq \mathrm{B}(\Gamma)$ *, and*
   (b) *if $\varphi, \psi \in \mathrm{B}(\Gamma)$, then also $\neg\varphi$, $(\varphi \vee \psi) \in \mathrm{B}(\Gamma)$.*

Definition 8. *Let $t \geqslant 0$, $m \geqslant 1$. Let $\underline{A}$, $\underline{B}$ be two t-bit pictures of equal height m. With $\underline{AB}$ we denote the t-bit picture obtained by concatenating the first column of $\underline{B}$ to the rightmost column of $\underline{A}$. More precisely: If $\underline{A} = (\underline{[m, n^A]}, X_1^A, \ldots, X_t^A)$ and $\underline{B} = (\underline{[m, n^B]}, X_1^B, \ldots, X_t^B)$, then $\underline{AB} = (\underline{[m, n^A + n^B]}, X_1^{AB}, \ldots, X_t^{AB})$, where $X_i^{AB} := X_i^A \cup \{(\alpha, n^A + j) : (\alpha, j) \in X_i^B\}$ for each $i \leqslant t$.*

The following proposition provides a characterization of the (k,q,r)-game which enables us to prove Theorem 2.
In fact, the game was designed in such a way that these properties hold: Winning condition 1 is necessary to obtain part 1, winning conditions 2 and 3 are essentially needed to establish part 3 and thus part 4 of this proposition.

Proposition 2 (Properties of the (k,q,r)-game). *Let $k, q, r, t \geqslant 0$.*

1. *If $\underline{A} \;{}_{k,q,r}{\equiv}\; \underline{B}$ then $\underline{A}$ and $\underline{B}$ cannot be distinguished by formulas in $\mathrm{B}(\Sigma_k^{q,r})$, i.e. for all $\varphi \in \mathrm{B}(\Sigma_k^{q,r})$ we have $\underline{A} \models \varphi \iff \underline{B} \models \varphi$.*
2. *The number $e_k^{q,r,t}(m)$ of equivalence classes in the (k,q,r)-game on t-bit pictures of height m is at most $(k+1)$-fold exponential in m, i.e. there is a constant $c = c_{q,t+kr}$ (which does not depend on m) such that $e_k^{q,r,t}(m) \leqslant s_{k+1}(cm)$ for all $m \geqslant 1$ (i.e. $e_k^{q,r,t}(m) \leqslant s_{k+1}(\mathcal{O}(m))$).*
3. *Let $\underline{A}$, $\underline{B}$, $\underline{C}$ be t-bit pictures of equal height.*
   If $\underline{A} \;{}_{k,q,r}{\equiv}\; \underline{B}$ then $\underline{AC} \;{}_{k,q,r}{\equiv}\; \underline{BC}$, i.e. we can extend Duplicator's winning strategy on $\underline{A}$, $\underline{B}$ to a winning strategy even on $\underline{AC}$, $\underline{BC}$.
4. *To each t-bit picture $\underline{A}$ of height m there exists a t-bit picture $\underline{B}$ of height m and width $\leqslant e_k^{q,r,t}(m)$ such that $\underline{A} \;{}_{k,q,r}{\equiv}\; \underline{B}$.*

The parts 1 and 2 of Proposition 2 are proved by standard methods; part 3 is an application of Schwentick's Extension Theorem ([Schw96]), and part 4 follows directly from part 3. Before giving a proof for Proposition 2 we will first apply it to prove Theorem 2, i.e. to prove that Σ_k-definable functions are at most k-fold exponential.

Proof of Theorem 2
Let $k \geqslant 1$. Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be a function defined by a Σ_k-sentence φ, i.e for all $m, n \geqslant 1$ we have $\underline{[m,n]} \models \varphi \iff n = f(m)$.

Let $q, r, t \geqslant 0$ such that φ is of the form $\exists X_1, \ldots, X_t \psi$ with $\psi \in \Pi_{k-1}^{q,r}$.
For each $m \geqslant 1$ we will show that $f(m) \leqslant e_{k-1}^{q,r,t}(m)$. Part 2 of Proposition 2 then implies that $f(m) \leqslant s_k(\mathcal{O}(m))$, i.e. that f is at most k-fold exponential.

As $\underline{[m, f(m)]} \models \varphi$, there exist sets $X_1^A, \ldots, X_t^A \subseteq [m, f(m)]$ such that $\underline{A} := (\underline{[m, f(m)]}, X_1^A, \ldots, X_t^A) \models \psi$. By part 4 of Proposition 2 there exists a t-bit picture $\underline{B} = (\underline{[m,n]}, X_1^B, \ldots, X_t^B)$ of width $n \leqslant e_{k-1}^{q,r,t}(m)$ such that $\underline{A} \;{}_{k-1,q,r}\!\!\equiv\; \underline{B}$. By part 1 of Proposition 2, $\underline{A}$ and $\underline{B}$ cannot be distinguished by $\Pi_{k-1}^{q,r}$-formulas. Thus $\underline{B} \models \psi$, and $\underline{[m,n]} \models \exists X_1, \ldots, X_t \psi$, i.e. $\underline{[m,n]} \models \varphi$. As φ defines the function f, we conclude that $n = f(m)$ and thus $f(m) \leqslant e_{k-1}^{q,r,t}(m)$. □

Proving Proposition 2

For convenience in analysing the (k, q, r)-game, we define the following sub-game:

The q-round FO game on two t-bit pictures $\underline{A}$, $\underline{B}$
consists of q first-order rounds. Let x_j^A (resp. x_j^B) be the vertex chosen in $\underline{A}$ (resp. $\underline{B}$) in the jth first-order round.
Duplicator wins iff the following two conditions hold:

1. Considering only the vertices $x_1^A, \ldots, x_q^A$ and $x_1^B, \ldots, x_q^B$, the pictures $\underline{A}$ and $\underline{B}$ look the same.
2. On the last 2^q columns, $x_1^A, \ldots, x_q^A$ and $x_1^B, \ldots, x_q^B$ are column-consistent.

We write $\underline{A} \equiv_q \underline{B}$ iff Duplicator has a winning strategy in the q-round FO game on $\underline{A}$ and $\underline{B}$.
Accordingly, for $p \in \{0, \ldots, q\}$ we write $(\underline{A}, x_1^A, \ldots, x_p^A) \equiv_{q-p} (\underline{B}, x_1^B, \ldots, x_p^B)$ iff Duplicator has a winning strategy after p rounds in which the vertices $x_1^A, \ldots, x_p^A$ and $x_1^B, \ldots, x_p^B$ were chosen.

Proof of Part 1 of Proposition 2:
This is a direct consequence of the "easy part" of Ehrenfeucht's Theorem (cf. [EF95], Theorem 1.2.8) and can be proved in the following way:
By induction on the structure of the formulas one first shows that if $\underline{A} \equiv_q \underline{B}$ then $\underline{A}$ and $\underline{B}$ cannot be distinguished by FO^q-formulas. Applying this, one obtains (again, by induction on the structure of the formulas) that if $\underline{A} \;{}_{k,q,r}\!\!\equiv\; \underline{B}$ then $\underline{A}$ and $\underline{B}$ cannot be distinguished by $\mathrm{B}(\Sigma_k^{q,r})$-formulas. □

To prove part 2 of Proposition 2, we have to count the number of equivalence classes. The following "*game-types*" are a useful tool to characterize equivalence classes.

First, we define game-types for the q-round FO game:
Let $t, q \geqslant 0$, $\underline{A}$ a t-bit picture of width $n \geqslant 1$. Let $x_1^A, \ldots, x_q^A \in A$, and let $\beta_j \geqslant 0$ such that x_j^A is in column $(n - \beta_j)$. For each $j \in \{1, \ldots, q\}$ we define

$$c_j \;:=\; \min\{\beta_j, 2^q\} \;\in\; \{0, \ldots, 2^q\}.$$

Inductively we define the following game-types for all $p \in \{0, \ldots, q\}$:

$$\begin{aligned}\mathrm{type}_q^q(\underline{A}, x_1^A, \ldots, x_q^A) \;:=\; (\,& (c_1, \ldots, c_q), \\ & \{\, X_i x_j \;:\; x_j^A \in X_i^A,\ i \in \{1, \ldots, t\},\ j \in \{1, \ldots, q\} \,\}, \\ & \{\, S_i x_j x_{j'} \;:\; (x_j^A, x_{j'}^A) \in S_i^A,\ i \in \{1, 2\},\ j, j' \in \{1, \ldots, q\} \,\}, \\ & \{\, x_j = x_{j'} \;:\; x_j^A = x_{j'}^A,\ j, j' \in \{1, \ldots, q\} \,\} \;)\end{aligned}$$

$\text{type}_q^p(\underline{A}, x_1^A, .., x_p^A) := \{ \text{type}_q^{p+1}(\underline{A}, x_1^A, .., x_p^A, x_{p+1}^A) : x_{p+1}^A \in A \}$.

In particular, for $p=0$ we define $\text{type}_q^{\text{FO}}(\underline{A}) := \text{type}_q^0(\underline{A})$.

Lemma 1. $\underline{A} \equiv_q \underline{B} \iff \text{type}_q^{\text{FO}}(\underline{A}) = \text{type}_q^{\text{FO}}(\underline{B})$
(for all $t, q \geqslant 0$ and all t-bit pictures $\underline{A}$ and $\underline{B}$).

Proof. By induction on $q-p$ one can easily show for all $p \in \{0, .., q\}$ that

$$(\underline{A}, x_1^A, .., x_p^A) \equiv_{q-p} (\underline{B}, x_1^B, .., x_p^B)$$
$$\iff \text{type}_q^p(\underline{A}, x_1^A, .., x_p^A) = \text{type}_q^p(\underline{B}, x_1^B, .., x_p^B). \qquad \square$$

Next we define game-types for the (k,q,r)-game:
Let $t \geqslant 0$ and $\underline{A}$ a t-bit picture, i.e. $\underline{A} = (\underline{[m,n]}, X_1^A, .., X_t^A)$, where $m, n \geqslant 1$ and $X_1^A, .., X_t^A \subseteq [m,n]$.

With $\underline{A}_{\lfloor 2^q}$ we denote the picture consisting of the last 2^q columns of $\underline{A}$, i.e.

$\underline{A}_{\lfloor 2^q} := (\underline{[m, 2^q]}, X_1', .., X_t')$, where
$X_i' := \{ (\alpha, j) : j \in \{1, .., 2^q\}, (\alpha, n-2^q+j) \in X_i^A \}$ for all $i \leqslant t$.

Let $k, q, r \geqslant 0$. Inductively we define

$\text{type}_0^{q,r}(\underline{A}) := (\text{type}_q^{\text{FO}}(\underline{A}), \underline{A}_{\lfloor 2^q})$
$\text{type}_k^{q,r}(\underline{A}) := \{ \text{type}_{k-1}^{q,r}(\underline{A}, X_{t+1}^A, .., X_{t+r}^A) : X_{t+1}^A, .., X_{t+r}^A \subseteq A \}$.

Lemma 2. $\underline{A} \equiv_{k,q,r} \underline{B} \iff \text{type}_k^{q,r}(\underline{A}) = \text{type}_k^{q,r}(\underline{B})$
(for all $k, q, r, t \geqslant 0$ and all t-bit pictures $\underline{A}$ and $\underline{B}$).

Proof. By induction on k; for $k=0$ apply Lemma 1. $\square$

For all $k, q, r, t \geqslant 0$ and $m \geqslant 1$ we define

$$E_k^{q,r,t}(m) := \{ \text{type}_k^{q,r}(\underline{A}) : \underline{A} \text{ is a } t\text{-bit picture of height } m \}.$$

From Lemma 2 we know that $e_k^{q,r,t}(m) = |E_k^{q,r,t}(m)|$, i.e. the number $e_k^{q,r,t}(m)$ of equivalence classes in the (k,q,r)-game on t-bit pictures of height m is exactly the number of the corresponding game-types.

For all $p \in \{0, .., q\}$ we define

$T_{q,t}^p := \{ \text{type}_q^p(\underline{A}, x_1^A, .., x_p^A) : \underline{A} \in t\text{-pic}, x_1^A, .., x_p^A \in A \}$
$T_{q,t}^{\text{FO}} := T_{q,t}^0 = \{ \text{type}_q^{\text{FO}}(\underline{A}) : \underline{A} \in t\text{-pic} \}$
$c_{q,t}^{\text{FO}} := |T_{q,t}^{\text{FO}}| =$ the number of different types (i.e. equivalence classes) in the q-round FO game on t-bit pictures.

Lemma 3. $c_{q,t}^{\text{FO}} < \infty$; *more precisely:*[3] $c_{q,t}^{\text{FO}} \leqslant s_q(2^{4q^2+tq+q})$ *(for all $q, t \geqslant 0$).*

Proof. By induction on $q-p$ we show for all $p \in \{0, .., q\}$ that

$$|T_{q,t}^p| \leqslant s_{q-p}(2^{4q^2+tq+q}).$$

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[3] Recall that s_q is the q-fold exponential function introduced in Definition 5.

$\underline{p=q}:\quad |T^q_{q,t}| \overset{\text{def}}{\leqslant} |\{0,..,2^q\}^q \times 2^{\{1,..,t\}\times\{1,..,q\}} \times 2^{\{1,2\}\times\{1,..,q\}^2} \times 2^{\{1,..,q\}^2}|$
$= (2^q+1)^q \cdot 2^{tq} \cdot 2^{2q^2} \cdot 2^{q^2} \leqslant (2^{q+1})^q \cdot 2^{3q^2+tq} = 2^{4q^2+tq+q}$.

$\underline{p{+}1 \mapsto p}$: As each element in $T^p_{q,t}$ is a subset of $T^{p+1}_{q,t}$, we have $|T^p_{q,t}| \leqslant 2^{|T^{p+1}_{q,t}|}$, and hence $|T^p_{q,t}| \leqslant s_{q-p}(2^{4q^2+tq+q})$. □

Proof of Part 2 of Proposition 2:
By induction on k we show that $e^{q,r,t}_k(m) \leqslant s_k(c^{\text{FO}}_{q,t+kr} \cdot 2^{(t+kr)2^q m})$.

$\underline{k=0}:\quad e^{q,r,t}_0(m) = |E^{q,r,t}_0(m)|$
$\overset{\text{def}}{=} |\{ (\text{type}^{\text{FO}}_q(\underline{A}), \underline{A}_{\lfloor 2^q}) : \underline{A} \text{ is a } t\text{-bit picture of height } m \}|$
$\leqslant |\{ \text{type}^{\text{FO}}_q(\underline{A}) : \underline{A} \in t\text{-pic} \}| \cdot |\{ (\underline{[m,2^q]}, X_1,..,X_t) : X_1,..,X_t \subseteq [m,2^q] \}|$
$= c^{\text{FO}}_{q,t} \cdot \left(2^{m2^q}\right)^t = c^{\text{FO}}_{q,t} \cdot 2^{t2^q m}$

$\underline{k{-}1 \mapsto k}$: As each element in $E^{q,r,t}_k(m)$ is a subset of $E^{q,r,t+r}_{k-1}(m)$, we obtain that $|E^{q,r,t}_k(m)| \leqslant 2^{|E^{q,r,t+r}_{k-1}|}$, and hence $e^{q,r,t}_k(m) \leqslant s_k(c^{\text{FO}}_{q,t+kr} \cdot 2^{(t+kr)2^q m})$.
If we define $c := c_{q,t+kr} := (\log_2 c^{\text{FO}}_{q,t+kr}) + (t+kr)2^q$, we obtain that $e^{q,r,t}_k(m) \leqslant s_k(2^{cm})$ for all $m \geqslant 1$, and thus $e^{q,r,t}_k(m) \leqslant s_{k+1}(\mathcal{O}(m))$. □

To prove part 3 of Proposition 2, i.e. to extend Duplicator's winning strategy on $\underline{A}$ and $\underline{B}$ to a winning strategy even on $\underline{AC}$ and $\underline{BC}$, we play the first-order rounds of the (k,q,r)-game according to Schwentick's Extension Theorem.

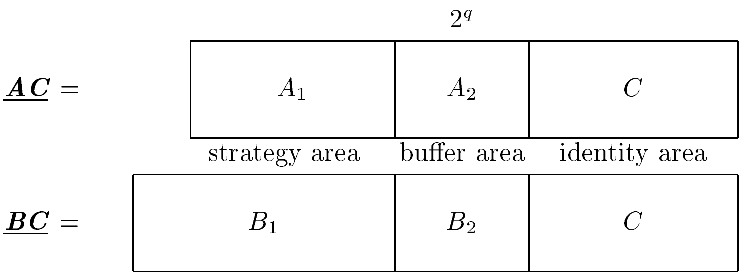
Proof of Part 3 of Proposition 2:
Let $\underline{A}$, $\underline{B}$, $\underline{C}$ be t-bit pictures of equal height m and of width n^A, n^B, n^C, respectively. Let $\underline{A} \underset{k,q,r}{\equiv} \underline{B}$, i.e. Duplicator has a winning strategy in the (k,q,r)-game on $\underline{A}$, $\underline{B}$. Duplicator's winning strategy on $\underline{AC}$, $\underline{BC}$ works as follows:

Case 1: $n^A \leqslant 2^q$ or $n^B \leqslant 2^q$.
As $\underline{A} \underset{k,q,r}{\equiv} \underline{B}$, we know that Duplicator can play column-consistent on the last 2^q columns. From this we conclude that $n^A = n^B$; and from winning condition 3 we obtain that $\underline{A} = \underline{B}$. Thus we have $\underline{AC} = \underline{BC}$, and in particular $\underline{AC} \underset{k,q,r}{\equiv} \underline{BC}$.

Case 2: $n^A > 2^q$ and $n^B > 2^q$.
In the second-order rounds, on the $\underline{C}$-part of $\underline{AC}$, $\underline{BC}$, Duplicator chooses the same colouring as Spoiler; on the $\underline{A}$-part and the $\underline{B}$-part of $\underline{AC}$, $\underline{BC}$ Duplicator colours according to her winning strategy on $\underline{A}$, $\underline{B}$.

Playing the first-order rounds according to Schwentick's Extension Theorem ([Schw96]), Duplicator directly obtains the following winning strategy on $\underline{AC}$, $\underline{BC}$: Let $\underline{A} = \underline{A_1}\underline{A_2}$ and $\underline{B} = \underline{B_1}\underline{B_2}$, where $\underline{A_2}$ (and $\underline{B_2}$, respectively) consists of the last 2^q columns of $\underline{A}$ (and $\underline{B}$, respectively).
Note that winning condition 3 ensures that $\underline{A_2}$ and $\underline{B_2}$ are coloured identically. Thus on $\underline{A_2C}$, $\underline{B_2C}$ Duplicator can win by simply choosing the same vertex as Spoiler did. On the other hand, on $\underline{A_1A_2}$, $\underline{B_1B_2}$ Duplicator can win according to her strategy on $\underline{A}$, $\underline{B}$, and winning condition 2 asserts that Duplicator can even play column-consistent on $\underline{A_2}$, $\underline{B_2}$.

At the beginning of the game we view $\underline{A_1}$ and $\underline{B_1}$ as the *strategy area*, $\underline{C}$ as the *identity area* and $\underline{A_2}$ and $\underline{B_2}$ as the *buffer area*.



Every time Spoiler chooses a vertex x in the *buffer area*, the three disjoint areas are modified:

- If x is closer to the *identity area* (with respect to the number of columns between x and this area), then the *identity area* is extended by all columns lying in the righthand half of the *buffer area*.
- If x is closer to the *strategy area*, then the *strategy area* is extended by all columns lying in the lefthand half of the *buffer area*.

After this modification Spoiler's vertex lies either in the *strategy area* or in the *identity area*. In the *identity area* Duplicator responds with the identity, i.e. she chooses the vertex in the same row with the same distance from the rightmost column as Spoiler's vertex; in the *strategy area* Duplicator responds according to her strategy on $\underline{A}$, $\underline{B}$.
One can easily see that after the pth first-order round the width of the *buffer area* is $\geqslant 2^{q-p}$. In particular, after q first-order rounds no vertex chosen in the *strategy area* is adjacent to any vertex chosen in the *identity area*. Thus one can easily see that winning condition 1 is satisfied. As obviously winning conditions 2 and 3 are also satisfied, we have found a winning strategy for Duplicator in the (k, q, r)-game on $\underline{AC}$, $\underline{BC}$, i.e. $\underline{AC} \equiv_{k,q,r} \underline{BC}$. □
A more detailed exposition is given in the appendix.

Applying part 3 of Proposition 2, we can now easily prove part 4 of that proposition, i.e. we show that for each t-bit picture $\underline{A}$ of height m there is a t-bit picture $\underline{B}$ of height m and width $\leqslant e_k^{q,r,t}(m)$ such that $\underline{A} \equiv_{k,q,r} \underline{B}$.

Proof of of Part 4 of Proposition 2:
Let $t \geqslant 0$, and $\underline{A}$ a t-bit picture of height m. We choose a t-bit picture $\underline{B}$ of height m and *minimal width* such that $\underline{A} \equiv_{k,q,r} \underline{B}$.
We consider the prefixes of $\underline{B}$ and make use of part 3 of Proposition 2.
If $\underline{B}$ has width $> e_k^{q,r,t}(m)$, then there must be two distinct prefixes of $\underline{B}$ lying in the same equivalence class, i.e. we can divide $\underline{B}$ into three parts $\underline{B_1}$, $\underline{B_2}$, $\underline{B_3}$ (where $\underline{B_3}$ might possibly be empty) such that $\underline{B} = \underline{B_1 B_2 B_3}$, $\underline{B_2}$ has width $\geqslant 1$ and $\underline{B_1} \equiv_{k,q,r} \underline{B_1 B_2}$.
By part 3 of Proposition 2 we obtain that $\underline{B_1 B_3} \equiv_{k,q,r} \underline{B_1 B_2 B_3}$.

Hence $\underline{B_1B_3} \underset{k,q,r}{\equiv} \underline{B} \underset{k,q,r}{\equiv} \underline{A}$, and $\underline{B_1B_3}$ has smaller width than $\underline{B}$, contradicting the minimality of $\underline{B}$. Thus, the width of $\underline{B}$ must be $\leqslant e_k^{q,r,t}(m)$. □

6 Some Final Remarks

In the previous section we have given a new, self-contained proof of Theorem 2.

In [MT97] Matz and Thomas were able to strengthen their upper bound on the growth rate to be valid even for functions definable by boolean combinations of Σ_k-formulas. Let us mention that, following their reasoning, this result can be established in our game-theoretic setting, too. Together with Proposition 1 one thus obtains a finer structure of the monadic hierarchy.

It might be of interest also to consider other kinds of grids, e.g. grids with built-in row- or column-*order* relations which are the transitive closure of the successor relations S_1 and S_2, respectively.

Our game-theoretical proof of Theorem 2 can directly be transferred to $\leqslant_1$-S_2-grids which have built-in row-order and column-successor relations. Together with Proposition 1 (which is also valid in the presence of order- instead of successor-relations) we hence obtain the strictness of the monadic hierarchy over $\leqslant_1$-S_2-grids.

On the other hand, for every k, the k-fold exponential function f_k (introduced in Proposition 1) is Σ_1-definable over $\leqslant_1$-$\leqslant_2$-grids (see [Schw97]). This implies that Theorem 2 is not valid over grids with built-in row- *and* column-order relations.
The infinity (resp. strictness) of the monadic hierarchy over (coloured or uncoloured) $\leqslant_1$-$\leqslant_2$-grids still is an open problem. But as Theorem 1 can directly be transferred to $\leqslant_1$-$\leqslant_2$-grids, the monadic hierarchy over coloured $\leqslant_1$-$\leqslant_2$-grids is strict unless the polynomial time hierarchy collapses.

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Appendix

Detailed proof of Part 3 of Proposition 2:
For $\underline{D} \in \{\underline{A}, \underline{B}\}$, let D be the domain of $\underline{D}$, and DC the domain of $\underline{DC}$. We define

$$\begin{aligned} D_1 &:= \{1,..,m\} \times \{1,..,n^D{-}2^q\} \\ D_2 &:= \{1,..,m\} \times \{n^D{-}2^q{+}1,..,n^D\} \\ D_C &:= \{1,..,m\} \times \{n^D{+}1,..,n^D{+}n^C\}, \end{aligned}$$

i.e. D_2 consists of the last 2^q columns of D, $D = D_1 \cup D_2$, and $DC = D \cup D_C$. By $n^{DC} := n^D{+}n^C$ we denote the width of $\underline{DC}$. For a set $X^{DC} \subseteq DC$ we define $X^D := X^{DC} \cap D$.

We define a bijective mapping $\kappa : A_2 \cup A_C \longrightarrow B_2 \cup B_C$, where $\kappa(\alpha, j) := (\alpha\, , n^{BC} - (n^{AC} - j))$; i.e. a vertex $x = (\alpha, j) \in A_2 \cup A_C$ in row α

with distance $(n^{AC}-j)$ from the rightmost column of $\underline{AC}$ is mapped onto the vertex $\kappa(x) \in B_2 \cup B_C$ in row α and with the same distance $(n^{AC}-j)$ from the rightmost column of $\underline{BC}$.

Claim 1: Duplicator can play the second-order rounds in such a way that for all $i \in \{0,..,k\}$, after round i the following holds: Let $t_i := t+ir$, and let $X^{AC}_{t+1},..,X^{AC}_{t_i} \subseteq AC$ and $X^{BC}_{t+1},..,X^{BC}_{t_i} \subseteq BC$ be the sets chosen so far. Then we have

1. $(\underline{A}, X^A_{t+1},..,X^A_{t_i}) \equiv_{k-i,q,r} (\underline{B}, X^B_{t+1},..,X^B_{t_i})$, and
2. $\kappa(X^{AC}_j \cap (A_2 \cup A_C)) = X^{BC}_j \cap (B_2 \cup B_C)$ for all $j \in \{1,..,t_i\}$.

Proof. By induction on i. On the $\underline{C}$-part of $\underline{AC}$, $\underline{BC}$, Duplicator chooses the same colouring (via κ) as Spoiler; on the $\underline{A}$-part and the $\underline{B}$-part of $\underline{AC}$, $\underline{BC}$ Duplicator colours according to her winning strategy on $\underline{A}$, $\underline{B}$. □

Let Duplicator play according to Claim 1, and let $X^{AC}_{t+1},..,X^{AC}_{t+kr}$ be the sets chosen in $\underline{AC}$, and $X^{BC}_{t+1},..,X^{BC}_{t+kr}$ the sets chosen in $\underline{BC}$ during the k second-order rounds.

Claim 2: For all $j \in \{1,..,t+kr\}$ and $x, y \in A_2 \cup A_C$ we have

1. $x \in X^{AC}_j \iff \kappa(x) \in X^{BC}_j$
2. $x = y \iff \kappa(x) = \kappa(y)$
3. $(x,y) \in S^{AC}_i \iff (\kappa(x), \kappa(y)) \in S^{BC}_i$ for $i \in \{1,2\}$.

Proof. The first part follows directly from part 2 of Claim 1, the second and the third part follow from the definition of κ. □

For $\underline{D} \in \{\underline{A}, \underline{B}\}$, by $\underline{\boldsymbol{DC}}$ we denote the $(t+kr)$-bit picture obtained by adding the colours chosen in the second-order rounds, i.e. $\underline{\boldsymbol{DC}} := (\underline{DC}, X^{DC}_{t+1},..,X^{DC}_{t+kr})$; and $\underline{\boldsymbol{D}} := (\underline{D}, X^{D}_{t+1},..,X^{D}_{t+kr})$, where $X^D_j := X^{DC}_j \cap D$.

From part 1 of Claim 1 we conclude that $\underline{\boldsymbol{A}} \equiv_q \underline{\boldsymbol{B}}$, i.e. Duplicator has a winning strategy in the q-round FO game on $\underline{\boldsymbol{A}}$ and $\underline{\boldsymbol{B}}$. In the following we will describe how to extend this to a winning strategy on $\underline{\boldsymbol{AC}}$ and $\underline{\boldsymbol{BC}}$.

For each $p \in \{0,..,q\}$ we will define numbers $\mathtt{b}_p$ and $\mathtt{i}_p$ denoting the width of the buffer area and the identity area, respectively, after the pth first-order round. The corresponding areas are defined as follows:

$$\begin{aligned}
strat_p(\underline{\boldsymbol{DC}}) &:= \{\, (\alpha,j) \,:\, 1 \leqslant \alpha \leqslant m,\ 1 \leqslant j \leqslant n^{DC}-(\mathtt{i}_p+\mathtt{b}_p) \,\}\\
buff_p(\underline{\boldsymbol{DC}}) &:= \{\, (\alpha,j) \,:\, 1 \leqslant \alpha \leqslant m,\ n^{DC}-(\mathtt{i}_p+\mathtt{b}_p) < j \leqslant n^{DC}-\mathtt{i}_p \,\}\\
iden_p(\underline{\boldsymbol{DC}}) &:= \{\, (\alpha,j) \,:\, 1 \leqslant \alpha \leqslant m,\ n^{DC}-\mathtt{i}_p < j \leqslant n^{DC} \,\}
\end{aligned}$$

We refer to the vertices in

$$\left.\begin{array}{r} strat_p(\underline{\boldsymbol{AC}}) \text{ and } strat_p(\underline{\boldsymbol{BC}}) \\ buff_p(\underline{\boldsymbol{AC}}) \text{ and } buff_p(\underline{\boldsymbol{BC}}) \\ iden_p(\underline{\boldsymbol{AC}}) \text{ and } iden_p(\underline{\boldsymbol{BC}}) \end{array}\right\} \text{ as the } \left\{\begin{array}{l} \text{strategy area} \\ \text{buffer area} \\ \text{identity area.} \end{array}\right.$$

		$\mathtt{b}_0 = 2^q$	$\mathtt{i}_0 = n^C$
$\underline{\boldsymbol{A}}\boldsymbol{C} =$	A_1	A_2	A_C
	strategy area	buffer area	identity area
$\underline{\boldsymbol{B}}\boldsymbol{C} =$	B_1	B_2	B_C

We define $\mathtt{b}_0 \;:=\; 2^q\,,\quad \mathtt{i}_0 \;:=\; n^C$.
In each first-order round $p \in \{1,..,q\}$, after Spoiler has chosen a vertex $x_p = (\alpha, j)$, we choose $\mathtt{b}_p$ and $\mathtt{i}_p$, and thus modify the three areas:
Let $\underline{D} \in \{\underline{A}, \underline{B}\}$ such that x_p was chosen in $\underline{\boldsymbol{D}}\boldsymbol{C}$.

- If $x_p \in \mathit{strat}_{p-1}(\underline{\boldsymbol{D}}\boldsymbol{C}) \cup \mathit{iden}_{p-1}(\underline{\boldsymbol{D}}\boldsymbol{C})$, then
  $\mathtt{b}_p := \mathtt{b}_{p-1}$, and $\mathtt{i}_p := \mathtt{i}_{p-1}$ (i.e. the areas do not change).
- If $x_p \in \mathit{buff}_{p-1}(\underline{\boldsymbol{D}}\boldsymbol{C})$, then let $\tilde{\jmath} \in \{1,..,\mathtt{b}_{p-1}\}$ such that $j = n^{DC} - (\mathtt{i}_{p-1} + \mathtt{b}_{p-1}) + \tilde{\jmath}$.
  - If $\tilde{\jmath} > \frac{\mathtt{b}_{p-1}}{2}$ (i.e. x_p is closer to the identity area), then we extend the identity area, i.e. $\mathtt{b}_p := \frac{\mathtt{b}_{p-1}}{2}$, and $\mathtt{i}_p := \mathtt{i}_{p-1} + \mathtt{b}_p$.
  - If $\tilde{\jmath} \leqslant \frac{\mathtt{b}_{p-1}}{2}$ (i.e. x_p is closer to the strategy area), then we extend the strategy area, i.e. $\mathtt{b}_p := \frac{\mathtt{b}_{p-1}}{2}$, and $\mathtt{i}_p := \mathtt{i}_{p-1}$.

  In particular, now Spoiler's vertex is either in the identity area or in the strategy area, i.e. either $x_p \in \mathit{iden}_p(\underline{\boldsymbol{D}}\boldsymbol{C})$ or $x_p \in \mathit{strat}_p(\underline{\boldsymbol{D}}\boldsymbol{C})$.

One can easily verify the following

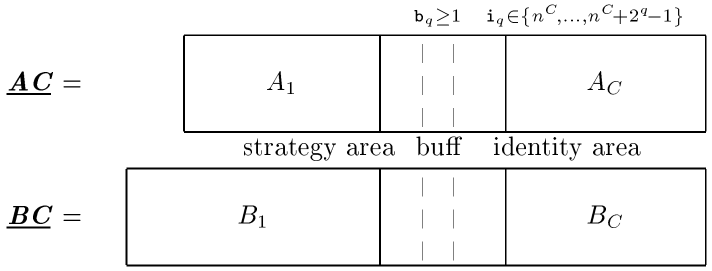
Claim 3: For all $p \in \{1,..,q\}$ and $\underline{D} \in \{\underline{A}, \underline{B}\}$ we have
1. $\mathtt{b}_p \geqslant 2^{q-p}$
2. $\emptyset \neq \mathit{buff}_p(\underline{\boldsymbol{D}}\boldsymbol{C}) \quad \subseteq \mathit{buff}_{p-1}(\underline{\boldsymbol{D}}\boldsymbol{C}) \subseteq D_2$
3. $D_1 \subseteq \mathit{strat}_{p-1}(\underline{\boldsymbol{D}}\boldsymbol{C}) \subseteq \mathit{strat}_p(\underline{\boldsymbol{D}}\boldsymbol{C}) \quad \subseteq D_1 \cup D_2 \;=\; D$
4. $D_C \subseteq \mathit{iden}_{p-1}(\underline{\boldsymbol{D}}\boldsymbol{C}) \subseteq \mathit{iden}_p(\underline{\boldsymbol{D}}\boldsymbol{C}) \quad \subseteq D_2 \cup D_C$.

Claim 4: Duplicator can play the first-order rounds in such a way that for every $p \in \{0,..,q\}$, after round p the following holds:
Let $x_1^{AC},..,x_p^{AC} \in AC$ and $x_1^{BC},..,x_p^{BC} \in BC$ be the vertices chosen so far. Then the following holds:

1. For every $i \in \{1,..,p\}$ we have
   (a) either $x_i^{AC} \in \mathit{strat}_p(\underline{\boldsymbol{A}}\boldsymbol{C})$ and $x_i^{BC} \in \mathit{strat}_p(\underline{\boldsymbol{B}}\boldsymbol{C})$
   (b) or, otherwise, $x_i^{AC} \in \mathit{iden}_p(\underline{\boldsymbol{A}}\boldsymbol{C})$ and $x_i^{BC} = \kappa(x_i^{AC}) \in \mathit{iden}_p(\underline{\boldsymbol{B}}\boldsymbol{C})$.
2. Let $\{1,..,p\} = \{i_1,..,i_l\} \cup \{j_1,..,j_{p-l}\}$ such that $x_{i_1}^{AC},..,x_{i_l}^{AC} \in \mathit{strat}_p(\underline{\boldsymbol{A}}\boldsymbol{C})$, and $x_{j_1}^{AC},..,x_{j_{p-l}}^{AC} \in \mathit{iden}_p(\underline{\boldsymbol{A}}\boldsymbol{C})$.
   Then we have $(\underline{\boldsymbol{A}}, x_{i_1}^{AC},..,x_{i_l}^{AC}) \equiv_{q-p} (\underline{\boldsymbol{B}}, x_{i_1}^{BC},..,x_{i_l}^{BC})$.

Proof. By induction on p. For $p = 0$ apply Claim 1. For the induction apply Claim 3 and make use of the column-consistency. In the strategy area Duplicator responds according to her strategy on $\underline{\boldsymbol{A}}, \underline{\boldsymbol{B}}$. In the identity area Duplicator responds according to the mapping κ. $\square$

Let Duplicator play according to Claim 4, and let $x_1^{AC}, .., x_q^{AC}$ be the vertices chosen in $\underline{\boldsymbol{AC}}$, and $x_1^{BC}, .., x_q^{BC}$ the vertices chosen in $\underline{\boldsymbol{BC}}$ during the q first-order rounds. Now we have the following situation:



From Claim 4 we know that $x_p^{DC} \notin \mathit{buff}_q(\underline{\boldsymbol{DC}})$ for all $p \in \{1, .., q\}$ and $\underline{D} \in \{\underline{A}, \underline{B}\}$. Let $\{1, .., q\} = \{i_1, .., i_l\} \cup \{j_1, .., j_{q-l}\}$ such that $x_{i_1}^{AC}, .., x_{i_l}^{AC} \in \mathit{strat}_q(\underline{\boldsymbol{AC}}) \subseteq A$, and $x_{j_1}^{AC}, .., x_{j_{q-l}}^{AC} \in \mathit{iden}_q(\underline{\boldsymbol{AC}}) \subseteq A_2 \cup A_C$. Claim 4 says that $x_{i_1}^{BC}, .., x_{i_l}^{BC} \in \mathit{strat}_q(\underline{\boldsymbol{BC}}) \subseteq B$, and $x_{j_1}^{BC}, .., x_{j_{q-l}}^{BC} \in \mathit{iden}_q(\underline{\boldsymbol{BC}}) \subseteq B_2 \cup B_C$.

Furthermore, from Claim 4 we know 1. $x_p^{BC} = \kappa(x_p^{AC})$ for all $p \in \{j_1, .., j_{q-l}\}$
2. $(\underline{\boldsymbol{A}}, x_1^{AC}, .., x_{i_l}^{AC}) \underset{q-q}{\equiv} (\underline{\boldsymbol{B}}, x_1^{BC}, .., x_{i_l}^{BC})$.

From this, together with Claim 2 we conclude that

- $x_p^{AC} \in X_j^{AC} \iff x_p^{BC} \in X_j^{BC}$ for all $j \leqslant t + kr$ and $p \in \{1, .., q\}$.
- As the areas are disjoint (i.e. $\mathit{strat}_q(\underline{\boldsymbol{DC}}) \cap \mathit{iden}_q(\underline{\boldsymbol{DC}}) = \emptyset$ for $\underline{D} \in \{\underline{A}, \underline{B}\}$), we have $x_p^{AC} \neq x_{p'}^{AC}$ and $x_p^{BC} \neq x_{p'}^{BC}$ for all $p \in \{i_1 .., i_l\}$ and $p' \in \{j_1 .., j_{q-l}\}$, and thus from 1. and 2. we obtain
  $x_p^{AC} = x_{p'}^{AC} \iff x_p^{BC} = x_{p'}^{BC}$ for all $p, p' \in \{1, .., q\}$.
- As the buffer area has width at least 1 (i.e. $\mathtt{b}_q \geqslant 1$), no vertex in the strategy area is adjacent (with respect to S_1 and S_2) to any vertex in the identity area, i.e. from 1. and 2. we conclude for all $p, p' \in \{1, .., q\}$ and $i \in \{1, 2\}$ that $(x_p^{AC}, x_{p'}^{AC}) \in S_i^{AC} \iff (x_p^{BC}, x_{p'}^{BC}) \in S_i^{BC}$.

Hence Duplicator's winning condition 1 is satisfied.
Furthermore, Duplicator played column-consistent on $A_2 \cup A_C$ and $B_2 \cup B_C$; and from Claim 1 we know that $A_2 \cup A_C$ and $B_2 \cup B_C$ are coloured identically, and thus the winning conditions 2 and 3 are satisfied. Hence Duplicator has a winning strategy in the (k, q, r)-game on $\underline{AC}$ and $\underline{BC}$, i.e. $\underline{AC} \underset{k,q,r}{\equiv} \underline{BC}$. $\square$