Translations, Reductions and Interpretations

J.A. Makowsky*
Department of Computer Science
Technion–Israel Institute of Technology, Haifa
e–mail: janos@cs.technion.ac.il

November 27, 1996

(Draft)

Abstract

We systematically study various notions of translations, interpretations and reductions which form part of the folklore of logic and show how they are relevant for computer science.

*partially supported by a Grant of the French-Israeli Binational Foundation and by the Fund for Promotion of Research of the Technion–Israeli Institute of Technology

Contents

1 Introduction

The topic of this chapter are translations of various logical concepts into one another. This concerns

- Translations of formulas of one formalism into another formalism
- Translations of structures of of a certain similarity type (vocabulary) into structures of another similarity type (vocabulary)

Such translations are commonly and ad hoc used in many instances in logic and mathematics, but the body of folklore techniques and specialized results is not readily available in textbooks or monographs. They concern

- consistency results, such as the existence of non–euclidean geometries or the relative consistency of Peano Arithmetic;
- undecidability results, such as Church's theorem;
- translation results, such as many–sorted logic is not more expressive than one–sorted logic;
- the reducibilty theory of classical recursion theory and complexity theory.

In general one studies functions between structures

$$f : Structures_1 \rightarrow Structures_2$$

or between finite combinatorial objects

$$g : Objects_2 \rightarrow Objects_1$$

where these objects may be formulas, strings, Turing machines and the like. In most cases f is a *semantic* translation, whereas g is a *syntactic* translation.

Consistency (relative consistency) results are now obtained along the following pattern: Let $\mathcal{A}$ be a structure for the language of geometry. We construct a (partial) function f which associates with $\mathcal{A}$ a structure $f(\mathcal{A})$ and show that the domain of f contains a model $\mathcal{B}$ of euclidean geometry and such that $f(\mathcal{B})$ is a model of geometry where the parallel axiom fails. Similarly, let $\mathcal{A}$ be a structure for the language of set theory. We construct a (partial) function f which associates with $\mathcal{A}$ a structure $f(\mathcal{A})$ in the language of arithmetic and show that for every $\mathcal{B}$ which is a model of some fragment of set theory $f(\mathcal{B})$ is a model of Peano arithmetic. In both cases we only use very little information about f, namely, that for $f : K_1 \rightarrow K_2$ a partial function we have that there is (for all) $\mathcal{A} \in K_1$ $f(\mathcal{A}) \in K_2$.

Undecidability results are very similar. Here $Objects_2$ are very often the set of Turing machines (register machines) M and $Objects_1$ a set of formulas F. One way to prove, say, Church's Theorem (the tautolgies of first order logic are not recursive) is to exhibit a function $g_1 : M \rightarrow F$ with the property that for $m \in M$ m halts on the empty input iff $g_1(m)$ is a tautology. In the proof of Trakhtenbrot's theorem (the set of first order formulas which have a finite model is not recursive) one exhibits a function $g_2 : M \rightarrow F$ with the property that for $m \in M$ m halts on the empty input iff $g_2(m)$ has a finite model.

In the case where these objects are formulas, $Structures_1$ and $Formulas_1$ are often linked by a *meaning function* $M(\phi, \mathcal{A})$ which has boolean values and gets the value 1 if the formula ϕ is true in the structure $\mathcal{A}$. This latter statement is denoted by $\mathcal{A} \models \phi$. The pair of functions f, g is then considered a *translation* if for every structure $\mathcal{A} \in Structures_1$ and formula $\phi \in Objects_2$ we have

$$\mathcal{A} \models g(\phi) \text{ iff } f(\mathcal{A}) \models \phi.$$

We shall be particularly interested in such pairs of functions which are induced by a *syntactically defined translation scheme.*

Translation results are usually of this type. To show that man–sorted and one–sorted first order logic are of the same expressive power one gives a pair of functions f, g. f converts many–sorted structures into one–sorted structures over an extended vocabulary, and g converts first order formulas over the extended vocabulary into many–sorted formulas over the original vocabulary.

In this chapter we embark on a systematic exposition of such translations and show how they are used and very different contexts.

Acknowqledgements

Dov Gabbay suggested to me that I write a chapter with the above title when neither him nor me had a clear idea what such a chapter should be about. During the years of writing I found exciting connections which I hope to share with the reader. I wish to thank Dov for suggesting work in this direction. I am also grateful to H.D. Ebbinghaus and J. Flum for fruitful discussions and showing me their supplement to the third edition of their book [EFT80] which deals with the same topic. I am also grateful to J. Rosicky for our conversation concerning category theory and translation schemes, and to E. Ravveh.

2 Prerequisites

We assume the reader is familiar with Logic and Model Theory as sketched in our chapter [Mak92] in volume of this Handbook Series. Nevertheless we list here what is needed and set up notation.

2.1 Logics

Logics
Model classes
etc

3 The General Framework

In this section we introduce the general framework for syntactically defined translation schemes and study their interesting properties. We also introduce a family of guiding examples. Finally, we introduce the notion abstract translation schemes and study under which conditions such abstract translation schemes are syntactically definable in a logic.

3.1 Translation Schemes

Definition 3.1 (Translation Schemes Φ) *Let τ_1 and τ_2 be two vocabularies and $\mathcal{L}$ be a regular logic. Let $\tau_2 = \{R_1, \ldots, R_m\}$ and let $\rho(R_i)$ be the arity of R_i. Let $\Phi = \langle \phi, \psi_1, \ldots, \psi_m \rangle$ be formulas of $\mathcal{L}(\tau_1)$. Φ is k–*feasible for τ_2 over τ_1 *if ϕ has exactly k distinct free variables and each ψ_i has $k\rho(R_i)$ distinct free variables. Such a $\Phi = \langle \phi, \psi_1, \ldots, \psi_m \rangle$ is also called a k–τ_1–τ_2–*translation scheme *or, shortly, a* translation scheme, *if the parameters are clear in the context.*

With a translation scheme Φ we can naturally associate a (partial) function Φ^* from τ_1–structures to τ_2–structures.

Definition 3.2 (The induced map Φ^*) *Let $\mathcal{A}$ be a τ_1–structure and Φ be k–feasible for τ_2 over τ_1. The structure $\mathcal{A}_\Phi$ is defined as follows:*

(i) The universe of $\mathcal{A}_\Phi$ is the set $A_\Phi = \{\bar{a} \in A^k : \mathcal{A} \models \phi(\bar{a})\}$;

(ii) the interpretation of R_i in $\mathcal{A}_\Phi$ is the set

$$\mathcal{A}_\Phi(R_i) = \{\bar{a} \in {A_\Phi}^{\rho(R_i)\cdot k} : \mathcal{A} \models \psi_i(\bar{a})\};$$

Note that $\mathcal{A}_\Phi$ is a τ_2–structure of cardinality at most $\mid A \mid^k$.

(iii) The the partial function $\Phi^* : Str(\tau_1) \to Str(\tau_2)$ is defined by $\Phi^*(\mathcal{A}) = \mathcal{A}_\Phi$. Note that $\Phi^*(\mathcal{A})$ is defined iff $\mathcal{A} \models \exists \bar{x}\phi$.

With a translation scheme Φ we can also naturally associate a function $\Phi^{\#}$ from $\mathcal{L}(\tau_2)$–formulas to $\mathcal{L}(\tau_1)$–formulas.

Definition 3.3 (The induced map $\Phi^{\#}$) *Let θ be a τ_2–formula and Φ be k–feasible for τ_2 over τ_1. The formula θ_Φ is defined inductively as follows:*

(i) For $R_i \in \tau_2$ and $\theta = R(x_1, \ldots, x_m)$ let $x_{j,h}$ be new variables with $i \leq m$ and $h \leq k$ and denote by $\bar{x}_i = \langle x_{i,1}, \ldots, x_{i,k} \rangle$. We put $\theta_\Phi = \psi_i(\bar{x}_1, \ldots, \bar{x}_m)$.

(ii) For the boolean connectives the translation distributes, i.e. if $\theta = (\theta_1 \vee \theta_2)$ then $\theta_\Phi = ({\theta_1}_\Phi \vee {\theta_1}_\Phi)$ and if $\theta = \neg\theta_1$ then $\theta_\Phi = \neg{\theta_1}_\Phi$, and similarly for $\wedge$.

(iii) *For the existential quantifier, we use relativization, i.e. if $\theta = \exists y \theta_1$, let $\bar{y} = \langle y_1, \ldots, y_k \rangle$ be new variables. We put*

$$\theta_\Phi = \exists \bar{y}(\phi(\bar{y}) \wedge \theta_{1\Phi}).$$

(iv) *The function $\Phi^\# : \mathcal{L}(\tau_2) \to \mathcal{L}(\tau_1)$ is defined by $\Phi^\#(\theta) = \theta_\Phi$.*

(v) *For Lindstrom quantifiers, we use relativized versions.*

To be completed

Proposition 3.4 *Let $\Phi = \langle \phi, \psi_1, \ldots, \psi_m \rangle$ be a k–τ_1–τ_2–translation scheme, $\mathcal{A}$ a τ_1-structure and θ a $\mathcal{L}(\tau_2)$–formula. Then $\mathcal{A} \models \Phi^\#(\theta)$ iff $\Phi^*(\mathcal{A}) \models \theta$.*

Proof: The proof is by induction on θ and follows from the definitions of Φ^* and $\Phi^\#$. □

To be completed

Exercise 3.5 *Reformulate this for quotient structures.*

3.2 Reductions

We first deal with the semantic notion of translating structures into structures.

Definition 3.6 (Reductions) *Let K_1, K_2 be classes of $\tau_1(\tau_2)$–structures closed under isomorphisms, $\mathcal{L}$ be a regular logic and Φ be a translation scheme.*

(i) *Φ^* is a* weak reduction *of K_1* to *K_2 if for every τ_1–structure $\mathcal{A}$ with $\mathcal{A} \in K_1$ also $\mathcal{A}_\Phi \in K_2$.*

(ii) *The translation Φ^* of K_1 to K_2 is* onto *if additionally for every $\mathcal{B} \in K_2$ there is an $\mathcal{A} \in K_1$ with $\mathcal{A}_\Phi$ isomorphic to $\mathcal{B}$.*

(iii) *Φ^* is a* reduction *of K_1* to *K_2 if for every τ_1–structure $\mathcal{A}$, $\mathcal{A} \in K_1$ iff $\mathcal{A}_\Phi \in K_2$.*

(iv) *We say that Φ* induces *a weak reduction (reduction) of K_1 to K_2, if Φ^* is a weak reduction (reduction) of K_1 to K_2. By abuse of language, we also say Φ* is a weak reduction (reduction) *instead of Φ induces a weak reduction (reduction).*

(v) *Let $k \in \mathbb{N}$. We say that K_1 is $\mathcal{L}$–k–*reducible *to K_2 ($K_1 \propto_{\mathcal{L}-k} K_2$), if there is a$\mathcal{L}$–$k$–translation scheme Φ for τ_2 over τ_1 such that Φ^* is a reduction from K_1 to K_2.*

(vi) *We say that K_1 is $\mathcal{L}$–*reducible *to K_2 ($K_1 \propto_{\mathcal{L}} K_2$), if $K_1 \propto_{\mathcal{L}-k} K_2$ for some $k \in \mathbb{N}$.*

Proposition 3.7 (Definability) *Let K_1, K_2 be classes of $\tau_1(\tau_2)$–structures closed under isomorphisms, $\mathcal{L}$ be a regular logic and Φ be a translation scheme in $\mathcal{L}$ such that Φ^* is a weak reduction from K_1 to K_2. Assume further that K_2 is definable, i.e. there is an $\mathcal{L}(\tau_2)$–formula θ such that $K_2 = Mod(\theta)$, and that either Φ^* is onto or a reduction. Then K_1 is definable by $\Phi^{\#}(\theta)$.*
Same for sets of sentences.

Better directly for sets ?

Proof: Assume that $\mathcal{A} \in K_1$. So $\mathcal{A}_\Phi \in K_2$ (weak reduction), hence $\mathcal{A}_\Phi \models \theta$. By Proposition 3.4 we get $\mathcal{A} \models \Phi^{\#}(\theta)$.
For the other direction we have two cases:
If Φ^* is a reduction and $\mathcal{A} \models \Phi^{\#}(\theta)$, then $\mathcal{A}_\Phi \models \theta$, by Proposition 3.4, and hence $\mathcal{A}_\Phi \in K_2$. As Φ is a reduction, we have $\mathcal{A} \in K_1$.
If Φ^* is onto and $\mathcal{A} \notin K_1$, $\mathcal{A}_\Phi \notin K_2$, hence $\mathcal{A}_\Phi \models \neg\theta$. By Proposition 3.4 we get $\mathcal{A} \models \Phi^{\#}(\neg\theta)$ and hence $\mathcal{A} \models \neg\Phi^{\#}(\theta)$.
For K_2 definable by a set of sentences Σ, we note that K_2 is the intersection of definable classes and apply the previous argument. □

If the logic is such, that the consequence relation is recursively enumerable, we can use a variation of the above proposition for decidability results.

Definition 3.8 (Decidable Theories) *Let $\mathcal{L}$ be a logic with recursive enumerable consequence relation. For a class K of τ–structures, we say that*

(i) K is L–decidable, *if*

$$Th_{\mathcal{L}}(K) = \{\theta \in \mathcal{L}(\tau) : \text{for every } \mathcal{A} \in K, \mathcal{A} \models \theta\}$$

is recursive.

(ii) We say that K is $\mathcal{L}$–elementarily–closed *if $\mathcal{A} \models Th_{\mathcal{L}}(\mathcal{A})$ iff $\mathcal{A} \in K$.*

(iii) We say that K is $\mathcal{L}$–closed *if $K = Mod_{\mathcal{L}}(Th_{\mathcal{L}}(K))$.*

Needed
? Closed under elementary equivalence
Needed ?

Proposition 3.9 (Rabin) *Let $\mathcal{L}$ be a logic with recursive enumerable consequence relation. Let K_1, K_2 be $\mathcal{L}$–closed classes of $\tau_1(\tau_2)$–structures and Φ be a translation scheme in $\mathcal{L}$.*

(i) If Φ^* is a weak reduction from K_1 to K_2 which is onto, K_1 is decidable, then K_2 is decidable. (Similar for K_1 decidable in TIME(f), SPACE(g) for suitable f and g, depending on the complexity of substitution in $\mathcal{L}$.)

(ii) If Φ^* is a translation from K_1 to K_2 which is onto and K_1 is decidable and K_2 definable by a single $\mathcal{L}$–sentence, then K_2 is decidable.

Proof: (i) Let $\theta \in \mathcal{L}(\tau_2)$. We want to checke whether $\theta \in Th(K_2)$. We check whether $\Phi^{\#}(\theta) \in Th(K_1)$. As $Th(K_1)$ is recursive, we get an answer.
If $\Phi^{\#}(\theta) \in Th(K_1)$, we have that $\theta \in Th(K_2)$, as Φ^* is a weak reduction.
If $\Phi^{\#}(\theta) \notin Th(K_1)$, there is $\mathcal{A} \in K_1$ with $\mathcal{A} \models \neg\Phi^{\#}(\theta)$. But as Φ^* is onto $\Phi^*(\mathcal{A}) \models \neg\theta$. As Φ^* is a weak reduction, $\Phi^*(\mathcal{A}) \in K_2$. Hence $\theta \notin Th(K_2)$.
(ii) Assume that $K_2 = Mod(\alpha)$. Now $\theta \in Th(K_2)$ iff $\alpha \rightarrow \theta$ is valid. As $\mathcal{L}$ has an r.e. consequence relation, $Th(K_2)$ is r.e.
On the other hand Now $\theta \notin Th(K_2)$ iff there is $\mathcal{B} \in K_2$ with $\mathcal{B} \models \neg\theta$ iff (here we use that Φ^* is onto) there is $\mathcal{A} \in K_1$ with $\mathcal{A} \models \Phi^{\#}(\neg\theta)$ iff $\Phi^{\#}(\theta) \notin Th(K_1)$.
This shows that $Th(K_2)$ is co–r.e. □

Examples 3.10 *(i) Non–definability*

(ii) Undecidability results

3.3 Interpretations

Free variables existentially bound

Definition 3.11 (Interpretations) *Let $\mathcal{L}$ be a regular logic and Σ_1, Σ_2 be sets of $\mathcal{L}(\tau_1)(\mathcal{L}(\tau_2))$–formulas and Φ be a (τ_1, τ_2)–k–translation scheme.*

(i) $\Phi^{\#}$ *is an* interpretation of Σ_2 in Σ_1 *if for every τ_2–formula θ with $\Sigma_2 \models \theta$ we have that $\Sigma_1 \models \Phi^{\#}(\theta)$.*

(ii) $\Phi^{\#}$ *is a* faithful interpretation of Σ_2 in Σ_1 *if for every τ_2–formula θ we have that $\Sigma_1 \models \Phi^{\#}(\theta)$ iff $\Sigma_2 \models \theta$.*

(iii) $\Phi^{\#}$ *is a* syntactic reduction of Σ_2 in Σ_1 *if for every τ_2–formula θ we have: If $\Sigma_2 \models \theta$ then $\Sigma_1 \models \Phi^{\#}(\theta)$ and if $\Sigma_2 \models \neg\theta$ then $\Sigma_1 \models \neg\Phi^{\#}(\theta)$.*

(iv) We say that Φ induces *a interpretation of Σ_2 to Σ_1, if $\Phi^{\#}$ is an interpretation of Σ_2 to Σ_1. By abuse of language, we also say Φ* is an interpretation *instead of Φ induces a interpretation.*

*(v) Let $k \in \mathbb{N}$. We say that Σ_2 is $\mathcal{L}$–k–*interpretable *in Σ_1 ($\Sigma_2 \propto_{\mathcal{L}-k} \Sigma_1$), if there is a k–translation scheme Φ for τ_2 over τ_1 such that $\Phi^{\#}$ is an interpretation of Σ_2 to Σ_1.*

*(vi) We say that Σ_1 is $\mathcal{L}$–*interpretable *in Σ_2 ($\Sigma_1 \propto_{\mathcal{L}} \Sigma_2$), if $\Sigma_1 \propto_{\mathcal{L}-k} \Sigma_2$ for some $k \in \mathbb{N}$.*

Lemma 3.12 *Let $\mathcal{L}$ be a regular logic and Σ_1, Σ_2 be sets of $\mathcal{L}(\tau_1)(\mathcal{L}(\tau_2))$–formulas and K_1, K_2 classes of τ_1–structures (τ_2–structures. Let Φ be a (τ_1, τ_2)–k–translation scheme.*

Not needed

(i) $\Phi^{\#}$ is an interpretation of Σ_2 in Σ_1 iff $\Phi^{\#}$ is also a syntactic reduction of Σ_2 to Σ_1.

(ii) If $\Phi^{\#}$ is an interpretation of Σ_2 in Σ_1 then Φ^* is a weak reduction from $Mod(\Sigma_1)$ to $Mod(\Sigma_2)$.

(iii) If Φ^* is a weak reduction of K_1 to K_2 then $\Phi^{\#}$ is an interpretation of $Th(K_2)$ into $Th(K_1)$.

Proof: (i) If $\Sigma_2 \models \theta$ then $\Sigma_1 \models \Phi^{\#}(\theta)$, by the definition of interpretations. On the other hand, and if $\Sigma_2 \models \neg\theta$ then $\Sigma_1 \models \Phi^{\#}(\neg\theta)$, by the definition of interpretations, and $\Sigma_1 \models \neg\Phi^{\#}(\theta)$, by the definition of $\Phi^{\#}$. The other direction is trivial.
(ii) Assume that $\mathcal{A} \models \Sigma_1$, but $\mathcal{B} = \Phi^*(\mathcal{A}) \not\models \Sigma_2$. So there is $\theta \in \Sigma_2$ such that $\mathcal{B} \not\models \theta$. By proposition 3.4 $\mathcal{A}) \not\models \Phi^{\#}(\theta)$ and hence $\mathcal{A}) \models \neg\Phi^{\#}(\theta)$ and $\Sigma_1 \not\models \Phi^{\#}(\theta)$. As $\Phi^{\#}$ is an interpretation, $\Sigma_2 \not\models \theta$, which contradicts $\theta \in \Sigma_2$.
(iii) Similar to (ii). □ To be completed ?

Proposition 3.13 (Decidability) *Let $\mathcal{L}$ be a regular logic and Σ_1, Σ_2 be sets of $\mathcal{L}(\tau_1)(\mathcal{L}(\tau_2))$–formulas and Φ be a (τ_1, τ_2)–k–translation scheme. If $\Phi^{\#}$ is a faithful interpretation of Σ_2 in Σ_1 and Σ_1 is decidable, so is Σ_2. Moreover, if Σ_1 is decidable in some complexity class* **C** *such that $\Phi^{\#}$ can be computed in* **C**, *so is Σ_2.*

Proof: Let $\theta \in \mathcal{L}(\tau_2)$. As Σ_1 is decidable and $\Phi^{\#}$ is computable, we check whether $\Sigma_1 \models \Phi^{\#}(\theta)$. By the definition of faithful inerpretations we have If $\Sigma_2 \models \theta$ iff If $\Sigma_1 \models \Phi^{\#}(\theta)$. □

Proposition 3.14 (Consistency) *Let $\mathcal{L}$ be a regular logic with a complete deduction system and Σ_1, Σ_2 be sets of $\mathcal{L}(\tau_1)(\mathcal{L}(\tau_2))$–formulas and Φ be a (τ_1, τ_2)–k–translation scheme. If $\Phi^{\#}$ is an interpretation of Σ_2 in Σ_1 and Σ_1 is consistent (satisfiable), so is Σ_2.*

Proof: Assume Σ_2 is inconsistent (unsatisfiable). So there is $\theta \in \mathcal{L}(tau_2)$ with $\Sigma_2 \models (\theta \wedge \neg\theta)$. Using lemma 3.12 (i) and the definition of $\Phi^{\#}$ we get that $\Sigma_1 \models \Phi^{\#}((\theta \wedge \neg\theta))$, which shows that Σ_1 is inconsistent (unsatisfiable). □

3.4 Interpretations vs. reductions

Let Φ be a translation scheme. Here we study the interplay of properties of Φ^* and $\Phi^{\#}$ which go beyond lemma 3.12.

Lemma 3.15 *Let Φ be a translation scheme such that Φ^* is a reduction from K_1 to K_2 which is onto. Then $\Phi^{\#}$ is a faithful interpretation of $Th_{\mathcal{L}}(K_2)$ to $Th_{\mathcal{L}}(K_1)$.*

Conversely ?

Proof: That $\Phi^{\#}$ is an interpretation follows from lemma 3.12. So assume $\theta \notin Th_{\mathcal{L}}(K_2)$. Hence there is $\mathcal{B} \in K_2$ with $\mathcal{B} \not\models \theta$ and, therefore $\mathcal{B} \models \neg\theta$. As Φ^* is onto, there is $calA \in K_1$ with $\Phi^*(\mathcal{A}) = \mathcal{B}$ and $\Phi^*(\mathcal{A}) \models \neg\theta$. Using proposition 3.4 we get that $\mathcal{A} \models \Phi^{\#}(\neg\theta)$, hence $\mathcal{A} \models \neg\Phi^{\#}(\theta)$ and $\mathcal{A} \not\models \Phi^{\#}(\theta)$. But from this we conclude that $\Phi^{\#}(\theta) \notin Th_{\mathcal{L}}(K_2)$, which completes the proof. □

4 A Museum of Examples I

In this section we present a first list of examples which illustrate the generality of the concept, as well as its precision.

4.1 Simple translation schemes

One of the simplest translations encountered in logic are the renaming of basic relations.

Example 4.1 (Renaming) *Let $\tau_1 = \{R_i : i \leq k\}$ and $\tau_2 = \{S_i : i \leq k\}$, with R_i and S_i of the same arity respectively. Let Φ be the (τ_1, τ_2) translation scheme given by $\Phi = \langle x = x; R_1(\bar{u}), \ldots, R_k(\bar{v})\rangle$. We call such a translation scheme, and also its induced maps Φ^* and $\Phi^{\#}$, a* renaming

In model theory one often looks at reducts of structures, e.g. the additive group of the reals, the order of the natural numbers. More generally we have a vocabulary τ_1 and a vocabulary $\tau_2 \subseteq \tau_1$. The reduct looks at a τ_1 structure forgetting all the relations but those mentioned in τ_2.

Example 4.2 (Reducts) *Given a vocabulary τ_1 and a vocabulary $\tau_2 = \{R_1, \ldots, R_k\} \subseteq \tau_1$. Let Φ be the (τ_1, τ_2) translation scheme given by $\Phi = \langle x = x; R_1(\bar{u}), \ldots, R_k(\bar{v})\rangle$. We call such a translation scheme, and also its induced maps Φ^* and $\Phi^{\#}$, a* reduct

The next example deals with conjunction.

Example 4.3 (Conjunction) *$k = 1$, $\tau_1 = \tau_2 = \{R(\bar{x})\}$ and $K_1 = Mod(\sigma_1)$ and $K_2 = Mod((\sigma_1 \vee \sigma_2))$.*
$\Phi = \langle (x = x \vee \sigma_2), R(\bar{x})\rangle$.
Φ^* is a partial function and a reduction.

So far, we have always used unvectorized translation schemes with without factorization. The next two examples show the power of vectorized translation schemes and factorization.

Example 4.4 (Cartesian power) *For simplicity, we assume that $k = 2$. $\tau_1 = \{R(\bar{x})\}$ with R binary and $\tau_2 = \{S(\bar{x})\}$ with S binary.*
$\Phi = \langle (x_1 = x_1 \vee x_2 = x_2), (R(x_3, x_4) \vee R(x_5, x_6))\rangle$.
It is easily seen that $\Phi^*(\mathcal{A})$ is isomorphic to the cartesian product $\mathcal{A}^2$.
The n–fold cartesian power is defined similarly.

4.2 Elimination of Function Symbols

Example 4.5 (Eliminating function symbols) *xxxx*

The next example is somewhat surprising. It is taken from [MPS90].

Example 4.6 *Let $\tau_1 = \emptyset$ and τ_2 consist of one constant symbol c. Let K_1 (K_2) consist of all infinite τ_1 (τ_2) structures. Let Φ be the (τ_1, τ_2) translation scheme given by $\Phi = \langle x = x \vee y = y; R_1(\bar{u}), \ldots, R_k(\bar{v}) \rangle$.* *to be completed*

Example 4.7 (Many–sorted structures) *Translating many–sorted into one–sorted.*

Example 4.8 (Disjoint union) *Translating many–sorted into one–sorted.*

4.3 Case Study I: Modelling Groups

Everybody mathematically educated is familiar with the concept of groups. However, textbooks in algebra usually differ in the presentation of groups as structures. The differences matter little, when one knows how to handle them. Our notions of translation schemes allow us to make this precise.

The simplest formalization is given as follows:

Example 4.9 (Groups 1) *τ_1 consists of one binary function symbol which we denote by M. The interpretation of M is usually denoted by $\cdot$, and is called* multiplication. *A group is a pair $\langle G, \cdot \rangle$ which satisfies the following:*

(i) (Associativity)

(ii) (Existence of inverse)

Example 4.10 (Groups 2) *τ_2 consists of one binary function symbol M and one unary function symbol I. The interpretation of M is usually denoted by $\cdot$, and is called* multiplication. *The interpretation of I is usually denoted by ${}^{-1}$, and is called* inverse. *A group is a triple $\langle G, \cdot, {}^{-1} \rangle$ which satisfies the following:*

(i) (inverse)

A group is a Multiplication alone, with inverse, with unit element We note that the notion of τ_1–substructure requires only closure under multiplication, whereus the notion of subgroup requires also closure under the inverse.

4.4 Case Study II: Databases

Example 4.11 (Database decomposition) *Lossless join decomposition.*

4.5 Counterexamples

Example 4.12 (Consistency of Peano Arithmetic) *Let K_1 be the models of ZFC (Zermelo–Fränkel Set Theory) and K_2 be the models of Peano Arithmetic. Here τ_1 consists of $\in$ only, whereas τ_2 has relation symbols for the order, addition, multiplication, 0 and 1.*

In ZFC the universe of the natural numbers is definable as the set of finite von–Neumann ordinals. The successor function and the order are the restriction of the ordinal successor function and order. Addition and multiplication are defined as one does in a first Calculus Course. Using this we can get a Φ such that, if $\mathcal{A} \models ZFC$, then $\Phi^*(\mathcal{A}) \models PA$. Furthermore, the axiom of choice is not needed to show this. Hence Φ^* is a weak reduction and $\Phi^\#$ is an interpretation.
Φ^* is not a reduction. Let $\mathcal{A} \models ZF$, but such that the axiom of choice is false. Hence $\mathcal{A} \notin K_1$. But still we have that $\Phi^*(\mathcal{A}) \models PA$.
$\Phi^\#$ is not a faithful interpretation. Let $CON(PA)$ be the formula of Peano Arithmetic which asserts that Peano Arithmetic is consistent. By Gödel's theorem we know that $CON(PA)$ is not a consequence of PA. But $\Phi^\#(CON(PA))$ is a consequence of ZFC.

From this we conclude:

Proposition 4.13 *There are weak reductions (interpretations) which are neither reductions nor faithful interpretations.*

Example 4.14 *We now modify example 4.12. Φ remains unchanged. Let CA be complete first order arithmetic, i.e. the complete theory of the standard model of arithmetic. Let K_1 be the models of $ZFC+\Phi^\#(CA)$ and and K_2 be the models of Peano Arithmetic.*
Clearly, by definition, $\Phi^\#$ is a faithful interpretation of PA into $ZFC+\Phi^\#(CA)$. On the other hand Φ^* is only a weak reduction, by propositio??, but not a reduction. There are $\mathcal{A} \notin K_1$, such that $\Phi^*(\mathcal{A}) \models CA$.

From this we conclude: Remark about onto

Proposition 4.15 *There are faithful interpretations which are weak reductions but not reductions.*

Example 4.16 *$\tau_1 = \tau_2 = \{R\}$ with R binary. Here we look at linear orders. K_1 consists of all linear orders with at least hundred elements and K_2 consists of all finite linear orders with at least two but less than hundred elements. Φ^* maps an order with less than hundred elements into its initial segment consisting of one element, and an order with at least hundred elements into its initial segment consisting of three elements. It is easy to see, how to define Φ.*
By definition Φ^* is a reduction which is not onto.
$\Phi^\#$ is an interpretation of $Th(K_2)$ into $Th(K_1)$, by lemma 3.12. But $\Phi^\#$ is not faithful. Let θ be the formula which says that there are at least three elements. Clearly $\theta \notin Th(K_2)$. But $\Phi^\#(\theta)$ says that, if an order has at least hundred elements, then the initial segement has at least three elements, which is true for K_1, hence $\Phi^\#(\theta) \in Th(K_1)$.

From this we conclude:

Proposition 4.17 *There are reductions (which are not onto) which are interpretations but not faithful.*

Our last counterexample is due to H.D. Ebbinghaus and concerns parametrized translation schemes.

Let τ_1 consist of one binary relation symbol R and consider the two τ_1–structures $\mathcal{Z}_< = \langle \mathbb{Z}, < \rangle$ and $\mathcal{N}_< = \langle \mathbb{N}, < \rangle$.

Proposition 4.18 *(i) There is a parametrized translation scheme Φ such that Φ^* is a reduction from $Th(\mathcal{Z}_<$ to $Th(\mathcal{N}_<$.* *check*

(ii) There is a no unparametrized translation scheme Φ such that Φ^* is a reduction from $Th(\mathcal{Z}_<$ to $Th(\mathcal{N}_<$ (even if we allow factorization). *check*

Proof: (i) Fix an element a of $\mathcal{Z}$. Now put $\Phi = \langle a < x; a < x \vee a < y \vee x < y \rangle$.
(ii) In $\mathcal{Z}$ every non trivial automorphisms moves all elements. In $\mathcal{N}$ there are non non trivial automorphisms. □ expand

5 Decidability and Undecidability

M. Rabin [Rab65] used translation schemes first in the early sixties for undecidability results. He simplified remarkably known results and used the method for new results as well. Extensive use of his method is also made in [?], part III. We shall exhibit a sequence of translation schemes deriving many known undecidability results from two basic results. Our exposition is, to a large extent, a reformulation of [?] and [Rab65].

5.1 The Basic Undecidable Theories

We shall use two classical undecidability results as our basis for reductions.

Our starting point is Goedel's classical result concerning the undecidability of number theory, which we now state in a form convenient for our purposes. Let us denote by $\mathcal{N}$ the τ_{arith}–structure $\mathcal{N} = \langle \mathbb{N}, +, \times, 0, 1 \rangle$. and by TA (True Arithmetic) the set of first order sentences true in $\mathcal{N}$.

Theorem 5.1 (Gödel) *TA is not recursively enumerable. More precisely, TA is Π^1_1–complete.*

The other result concerns a finitely axiomatizable subset Q of TA first isolated in [?]. Q consists of the logical consequences of the the following axioms:

$\forall v_0 \forall v_1 (suc(v_0) = suc(v_1) \rightarrow v_0 = v_1)$

$\forall v_0 \neg (suc(v_0) = 0)$

$\forall v_0 [\neg v_0 = 0) \rightarrow \exists v_1 (suc(v_1) = v_0)]$

$\forall v_0(v_0 + 0 = v_0)$

$\forall v_0 \forall v_1 [v_0 + suc(v_1) = suc(v_0 + v_1)]$

$\forall v_0(v_0 \cdot 0 = 0)$

$\forall v_0 \forall v_1 (v_0 \cdot suc(v_1) = v_0 \cdot v_1 + v_0)$

Theorem 5.2 *Q is not recursive. Moreover, if T is a consistent extension of Q, then T is inseparable.*

5.2 Undecidability via Q

We next consider the following theories. Let τ_{bin} be the vocabulary consisting of one binary relation symbol R.

S is the set of logical consequences of the following three axioms:

(Empty set)

(Extensionality)

(Pairs)

T_{graph} is the set of logical consequences of the following two axioms:

(Irreflexivity:) $\forall v_0 \neg R(v_0, v_0)$

(Symmetry:) $\forall v_0 \forall v_1 (R(v_0, v_1) \leftrightarrow R(v_1, v_0))$

Finally, let T_{bin} be the set formulas true in all structures with a binary relation.

Theorem 5.3 [?] *There is a first order translation scheme Φ_1 such ${\Phi_1}^{\#}$ is a translation and onto from $Mod(S)$ to $Mod(Q)$.* *Check onto*

[?] *There is a first order translation scheme Φ_2 such ${\Phi_2}^{\#}$ is a translation and onto from $Mod(T_{graph})$ to $Mod(S)$.*

[Rab65] *There is a first order translation scheme Φ_3 such ${\Phi_3}^{\#}$ is a translation and onto from $Mod(T_{bin})$ to $Mod(T_{graph})$. In fact, ${\Phi_3}^{\#}$ is also a translation and onto from $Mod_{fin}(T_{bin})$ to $Mod_{fin}(T_{graph})$.*

Corollary 5.4 (Monk) *S is essentially undecidable.*

(Kalmar, 1936) *The theory of a binary relation is undecidable (in fact finitely inseparable).*

(Church and Quine, 194?) *The theory of a irreflexivie, symmetric binary relation is undecidable (in fact finitely inseparable).*

5.3 Decidability of Successor Functions

We first look at the structure $\mathcal{N}_{suc} = \langle \mathbb{N}, suc, 0\rangle$ where suc is the unary successor function. We denote by $S1S$ the Monadic Second Order theory of $\mathcal{N}_{suc}$. Büchi showed that

Theorem 5.5 (Büchi) *$S1S$ is recursive. In fact it is* *Check Complexity*

This was generalized by Rabin. Let $\mathbf{T}_n$ the full n–ary tree and $\mathcal{T}_n$ the structure $\mathcal{T}_n\langle \mathbf{T}_n, suc_1, \ldots, suc_n\rangle$, where suc_i is the unary function mapping a node into its i–th successor. We denote by SnS the Monadic Second Order theory of $\mathcal{T}_n$.

Theorem 5.6 (Rabin) *SnS is recursive. In fact it is elementary...* *Check Complexity*

Rabin's theorem is proven via Automata Theory and is probably the most powerful decidability theorem proven sofar.

Theorem 5.7 *There is a MSOL translation scheme Φ from $S2S$ to boolean algebras with ideals.*

Corollary 5.8 *Boolean Algebras with ideals.*

Theorem 5.9 *There is a MSOL translation scheme Φ from $S2S$ to Presburger Arithmetic*

Corollary 5.10 *Pressburger Arithmetic*

Theorem 5.11 *There is a MSOL translation scheme Φ from $S2S$ to Abelian Groups.*

Theorem 5.12 *Abelian Groups.*

Theorem 5.13 *There is a MSOL translation scheme Φ from $S2S$ to Temporal Logic.* *Find exact Formulation*

Corollary 5.14 *Temporal Logics*

Problem 5.15 (Rabin) *Can one get the first order decidability of RCF via direct interpretations from the monadic second order decidability of SnS ?*

More generally, one might ask: A problem is complete for a complexity class via $\mathcal{L}$–interpreations if...... Is Sn complete for primitive recursive problems via MSOL–interpretations ?

5.4 Arithmetic without multiplication

We have seen before, that Presburger arithmetic is decidable for First Order Logic, cf. theorem ??, and that the theory of the successor is decidable for Monadic Second Order Logic, cf. Theorem ??. We now look at some variations.

Let $\mathcal{N}_{+,S}$ be defined as $\mathcal{N}_{+,S} = \langle \mathbb{N}, +, S, 0, 1 \rangle$ and let $\mathcal{N}_{suc,S}$ be defined as $\mathcal{N}_{suc,S} = \langle \mathbb{N}, suc, S, 0, 1 \rangle$ where S is any n–ary relation over $\mathbb{N}$.

We are interested in decidability questions for $\mathcal{N}_{+,S}$ and $\mathcal{N}_{suc,S}$ for various S both in First Order and Monadic Second Order Logic.

Of particular interest are the relations:

- SQ: binary, the squaring function $y = x^2$.
- D: binary, the divisibility relation x divides y.
- SQ_1: unary, the set of squares.
- P: unary, the set of primes

Proposition 5.16 *In $\mathcal{N}_{+,SQ}$ multiplication is First Order definable.*

Proof: $x \cdot y = z$ is defined by $\exists u(z + z = u \vee u + sq(x) + sq(x) = sq(x + y))$. □

Proposition 5.17 *In $\mathcal{N}_{+,D}$ squaring is First Order definable.*

Proof: We first note that the linear order $x \leq y$ is definable by $\exists z(x + z = y)$. Next, we observe that the least common multiple $lcm(x, y)$ and the greatest common divisor $gcd(x, y)$ are definable using divisibility and order. Now, for x, y with $gcd(x, y) = 1$ $x \cdot y = z$ iff $z = lcm(x, y)$. Using the fact that $gcd(x, x+1) = 1$ we can write $sq(x) = z$ iff $z + x = lcm(x, x + 1)$. □

Proposition 5.18 *In $\mathcal{N}_{+}$ divisibility is Second Order definable.*

Proof: $div(x, y)$ iff y is a multiple of x iff y is in the smallest set which contains x and is closed under the addition of x. □

From this we get that

Theorem 5.19 *The following theories are undecidable:*

(i) The First Order theories of $\mathcal{N}_{+,SQ}$ and $\mathcal{N}_{+,D}$.

(ii) The Monadic Second Order theory of Presburger arithmetic.

Corollary 5.20 *In $\mathcal{N}_{suc}$ addition is not Monadic Second Order definable.*

Proof: $S1S$ is decidable. If addition were definable in $S1S$, so would the Monadic Second Order theory of Presburger arithmetic, which contradicts the above theorem. □

The situation with the prime predicate is more delicate. The only known answers are under the assumption of the linear case of Schinzel's Hypothesis (LSH), which is a generalization of the Goldbach conjecture.

Theorem 5.21 (Bateman, Jockusch and Woods) *Assume LSH. Then multiplication is First Order definable in $\mathcal{N}_{+,P}$, and hence its First Order theory is undecidable.*

Surprisingly, we have the following

Theorem 5.22 (Bateman, Jockusch and Woods) *Assume LSH. Then the Monadic Second Order theory of $\mathcal{N}_{suc,P}$ is decidable. Hence, addition is not Monadic Second Order definable in $\mathcal{N}_{suc,P}$.*

5.5 The Real Numbers

Let us denote by $\mathcal{R}$ the structure $\mathcal{R} = \langle\, \mathbb{R}, +, \times, 0, 1\rangle$. and by RCF the set of first order sentences true in $\mathcal{R}$. In this subsection we illustrate the method of interpretations further for the study of decidability questions of variations of $\mathcal{R}$.

A. Tarski's fundamental result, [?] states that

Theorem 5.23 (Tarski) *RCF is recursive.*

His proof gives the axioms explicitely for RCF, ie. the axioms of Real Closed Fields, and proceeds to show that under these axioms elimination of quantifiers is possible.

There are now several natural questions concerning $\mathcal{R}$. The first which comes to our mind, is whether RCF_1, the set of sentences in Monadic Second Order Logic true in $\mathcal{R}$ is still recursive. The answer is no, and the proof is an illustration of the method of direct interpretations.

Proposition 5.24 (Folklore) *There is a translation scheme Φ in Monadic Second Order Logic such that for every model $\mathcal{A}$ of RCF $\Phi^{\#}(\mathcal{A})$ is isomorphic to $\mathcal{N}$. Hence, by theorem ??, RCF_1 is not recursively enumerable.*

Proof: We define in $\mathcal{A}$ the set of natural numbers as the smallest subset of the prime field containing 0 and which is closed under the operation $x + 1$. This is clearly expressible in Monadic Second Order Logic. Addition and multiplication are simply the restrictions to this set. □

Exercise 5.25 *Let $\mathcal{R}_+$ be the additive structure of $\mathcal{R}$, i.e. $\mathcal{R}_+ = \langle\, \mathbb{R}, +, 0, 1\rangle$. Clearly, the first order theory of $\mathcal{R}_+$ is decidable by Tarski's theorem. What can you say about the monadic second order theory of $\mathcal{R}_+$?*

Hint: *Clearly one can reduce the problem to the monadic second order theory of Presburger arithmetic. One can also define divisibility. There is a result about* $\langle \mathbb{N}, +, div, 0, 1\rangle$ *of which I do not recall the reference.*

The next set of questions concerns the structures $\mathcal{R}_{sin}\langle \mathbb{R}, +, \times, , sin, 0, 1\rangle$ and $\mathcal{R}_{exp}\langle \mathbb{R}, +, \times, exp, 0, 1\rangle$ where sin is the unary real function sinus and exp is the unary real function e^x. We denote by RCF_{sin} (RCF_{exp}) the set of first order sentences true in $\mathcal{R}_{sin}$ ($\mathcal{R}_{exp}$).

Again by the method of direct interpretation we get

Theorem 5.26 (Folklore) *There is a translation scheme* Φ *in First Order Order Logic such that for every model* $\mathcal{A}$ *of* RCF_{sin} $\Phi^{\#}(\mathcal{A})$ *is a model of* TA. *Hence, by theorem* **??**, RCF_{sin} *is not recursively enumerable.*

Proof: We first note that a number is positive iff it is a square. Hence the linear ordering is definable. Next we note that the number π is definable as the least positive zero of sin. With this we can define integer multiples of π, and hence the natural numbers. Again, the remaining operations are defined via restrictions. □

In contrast to this, Tarski conjectured that RCF_{exp} is decidable. In the last 20 years this conjecture has challenged many model theorists. Their joint effort has culminated in

Theorem 5.27 (Wilkie and xxxxx) RCF_{exp} *allows elimination of quantifiers and, assuming Shanuel's conjecture on exponential polynomials,* RCF_{exp} *is decidable.*

6 Translations and Definability

6.1 General Translations

Let $\mathcal{L}$ be a logic. Recall that for a class of τ–structures $K \subseteq Str(\tau)$ we define

$$Th_{\mathcal{L}}(K) = \{\phi \in \mathcal{L}(\tau) : \text{for every } \mathcal{A} \in K \mathcal{A} \models \phi\}.$$

Similarly, for $\Sigma \subseteq \mathcal{L}(\tau)$ we define

$$Mod_{\mathcal{L}}(\Sigma) = \{\mathcal{A} \in Str_{\mathcal{L}}(\tau) : \text{for every } \phi \in \Sigma \mathcal{A} \models \phi\}.$$

Definition 6.1 *Let*

$$f : Str_{\mathcal{L}}(\tau_1) \rightarrow Str_{\mathcal{L}}(\tau_2)$$

and

$$g : \mathcal{L}(\tau_2) \rightarrow \mathcal{L}(\tau_1)$$

be two functions. Let $K_1 \subseteq Str_{\mathcal{L}}(\tau_1)$ *and* $K_2 \subseteq Str_{\mathcal{L}}(\tau_2)$. *Let* $\Sigma_1 \subseteq \mathcal{L}(\tau_1)$ *and* $\Sigma_2 \subseteq \mathcal{L}(\tau_2)$.

(i) *f is a weak reduction (reduction, onto).*

(ii) *g is an interpretation (faithful interpretation, logical interpretation), if $g(\Sigma_2) \subseteq \Sigma_1$ ($\phi \in \Sigma_2$ iff $g(\phi) \in \Sigma_2$, if $\phi \in \Sigma_2$ then $g(\phi) \in \Sigma_2$, and $\neg\phi \in \Sigma_2$ then $\neg g(\phi) \in \Sigma_2$).*

(iii) *The pair $\langle f, g \rangle$ is an* abstract translation scheme *if for every $\mathcal{A} \in Str_{\mathcal{L}}(\tau_1)$ and for every $\phi \in \mathcal{L}(\tau_2)$ we have that*

$$\mathcal{A} \models g(\phi) \text{ iff } f(\mathcal{A}) \models (\phi)$$

(iv) *The pair $\langle f, g \rangle$ is an* abstract interpretation scheme *of K_1 into K_2, if additionally $f(K_1) \subseteq K_2$ (or, equivalently $f(Th_{\mathcal{L}}(K_2) \subseteq Th_{\mathcal{L}}(K_1))$.*

(v) *The pair $\langle f, g \rangle$ is an* abstract interpretation scheme *of Σ_2 into Σ_1, if additionally $g(\Sigma_2) \subseteq \Sigma_1$ (or, equivalently $g(Mod_{\mathcal{L}}(\Sigma_2) \subseteq Mod_{\mathcal{L}}(\Sigma_1))$.*

The notion of abstract translation scheme generalizes what we have studied so far. Most of the previous propositions and applications can be easily proven for the more abstract framework. For example we have: More examples ?

Proposition 6.2 *Let $\langle f, g \rangle$ be an abstract translation scheme.*

(i) *If f is a weak reduction from K_1 to K_2, then g is an interpretation of $Th_{\mathcal{L}}(K_2)$ to $Th_{\mathcal{L}}(K_1)$.*

(ii) *If g is an interpretation from Σ_2 to Σ_1, then f is a weak reduction of $Mod_{\mathcal{L}}(\Sigma_1)$ to $Mod_{\mathcal{L}}(\Sigma_2)$.*

Proposition 6.3 *Let Φ be a $\mathcal{L}$–translation scheme. Then the pair $\langle \Phi^*, \Phi^{\#} \rangle$ is an abstract translation scheme and all the other properties translate properly. Additionally, if $\Phi^{\#}$ is an interpretation of Σ_2 into Σ_1, then it is also a logical interpretation.*

The natural question arises, whether there are any other interesting abstract translation schemes. Our $\mathcal{L}$–translation schemes have the following additional properties:

Definition 6.4 *f is a* restructuring scheme *if for every $\mathcal{A} \in Str_{\mathcal{L}}(\tau_1)$ there is a unique expansion $\bar{A} \in Str_{\mathcal{L}}(\tau_1 \cup \tau_2)$ such that $\bar{A}\mid_{\tau_2} = f(\mathcal{A})$.*

Definition 6.5 *A* restructuring scheme f is definable in *calL if the class of $\tau_1 \cup \tau_2$–structures $\bar{A}$ such that $\bar{A}\mid_{\tau_2} = f(\mathcal{A})$ is definable.*

The purpose of the next section is to show for First Order Logic (and a more general class of logics) the definable restructuring schemes are exactly the explicit translation schemes. For this, we first discuss Beth's Theorem on implicit definitions.

6.2 Beth's Theorem

Let $\mathcal{L}$ be a logic, $\tau_1 = \tau_2 \cup \{R\}$ be vocabularies with $R \notin \tau_2$.

Definition 6.6 *Let $\Sigma_1 \subseteq \mathcal{L}(\tau_1)$, $\Sigma_2 \subseteq \mathcal{L}(\tau_2)$, $\phi(\bar{x}) \in \mathcal{L}(\tau_2)$ and R with the same arity as there are free variables in ϕ. $\Sigma_2 \cup \{\phi\}$ is an* explicit definition of R relative to Σ_1 *if*

$$\Sigma_1 \models \Sigma_2 \cup \{(\forall \bar{x}(\phi(\bar{x}) \to R(\bar{x}))\}$$

Definition 6.7 *Let $\Sigma_1 \subseteq \mathcal{L}(\tau_1)$, $\Sigma_2 \subseteq \mathcal{L}(\tau_2)$ and R with the same arity as there are free variables in ϕ.*

(i) *Σ_1 is an* implicit definition of R relative to Σ_2 *if for every $\mathcal{B} \models \Sigma_2$ there is at most one $\mathcal{A} \in \Sigma_1$ with $\mathcal{A}_{\tau_2} = \mathcal{B}$.*

(ii) *Σ_1 is a* weak implicit definition of R relative to Σ_2 *if for every $\mathcal{B} \models \Sigma_2$ there is exactly one $\mathcal{A} \in \Sigma_1$ with $\mathcal{A}_{\tau_2} = \mathcal{B}$.*

It is easy to see that if $\Sigma_2 \cup \{\phi\}$ is an explicit definition of R relative to Σ_1 then Σ_1 is an implicit definition of R. For first order logic one can prove the converse.

Theorem 6.8 (Beth) *Let $\Sigma_1 \subseteq \mathcal{L}(\tau_1)$, $\Sigma_2 \subseteq \mathcal{L}(\tau_2)$ and R with the same arity as there are free variables in ϕ. If Σ_1 is an implicit definition of R relative to Σ_2, then there is $\phi(\bar{x}) \in \mathcal{L}(\tau_2)$ such that $\Sigma_2 \cup \{\phi\}$ is an explicit definition of R relative to Σ_1.*

Next we want to rephrase these definitions with the help of translation schemes.

implicit definability: $K_2 = Mod(\Sigma)$. Every $\mathcal{B} \in K_2$ has a unique expansion $\mathcal{A} \models \phi(S)$. In other words: For Φ which forgets S there is a unique inverse of Φ^* from $K_1 = Mod(\Sigma \cup \{\phi\})$. Then this inverse is of the form Ψ^* for some Ψ. Can we generalize to arbitrary Φ ?

6.3 Invertible Reductions and Beth's Theorem

Expansions
Restrictions (Reductions)
Implicit definability
Beth's Theorem (traditional version)
Invertible translations, reductions
Weakly invertible ......
Inversion theorem (JAM)

7 Feferman–Vaught Theorem

8 Finite Model Theory and Complexity

9 Interpretability of theories

In this section we survey a very active aspect of translation schemes: interpretability of logical theories. Comprehensive surveys are [MPS90] and [Smo85]. I have drawn extensively from both these sources. We first look at various approaches.

9.1 Recursive Interpretability of Theories

The most general definition uses no translation scheme at all and looks only at the syntactic side.

Theories are sets of τ–sentences in first order logic. We denote theories by $T, T_1, T_2, \ldots$ Vocabularies are countable and have some recursive presentation. Theories are countable.

Definition 9.1 (Recursive Interpretations) *Let g be a recursive function from $SENT(\tau_1)$ to $SENT(\tau_2)$. g is a* recursive interpretation of T_1 in T_2 *if for every τ_1–formula θ we have:*
If $T_1 \models \theta$ then $T_2 \models g(\theta)$ and if $T_1 \models \neg\theta$ then $T_2 \models \neg g(\theta)$.

Definition 9.2 (Interpretations) *$\Phi^{\#}$ is a* syntactic reduction of Σ_2 in Σ_1 *if for every τ_2–formula θ we have:*
If $\Sigma_2 \models \theta$ then $\Sigma_1 \models \Phi^{\#}(\theta)$ and if $\Sigma_2 \models \neg\theta$ then $\Sigma_1 \models \neg\Phi^{\#}(\theta)$.

The reducibility degrees form a set. We can define a partial order

Proposition 9.3 *There is*

For finitely axiomatizable theories the reducibly degrees form a set. On this set we define a lattice structure L_0.

Theorem 9.4 (A. S. Stern) *L_0 is a distributive lattice.*

Instead we can consider recursive reducibility via functions of low complexity, such as LogSpace or Polynomial Time. Let **C** be any 'reasonably' low leven complexity class.

Problem 9.5 *Study the preorder among finitely axiomatizable theories via* **C***–reducibility.*

9.2 Interpretability Types

In the case of recursive reducibility we only required that the function g between formulas be recursive. Now we look at the case where g is of the form $\Phi^{\#}$ for some translation scheme Φ in first order logic FOL.

We consider here three definitions.

Definition 9.6 (Interpretability between theories) *Let T_1, T_2 be two first order theories over the vocabularies τ_1, τ_2 respectively.*

(i) We say that T_1 is globally interpretable *in T_2, and we write $T_1 \rightarrow_g T_2$, if there is a (τ_2, τ_1)–translation scheme Φ such that for every $\phi \in SENT(\tau_1)$ if $T_1 \models \phi$ then $T_2 \models \Phi^{\#}(\phi)$.*

(ii) We say that T_1 is globally interpretable *in T_2* with parameters*, and we write $T_1 \rightarrow_{gp} T_2$, if there is a (τ_2, τ_1)–translation scheme Φ with free variables $\bar{x}$ such that for every $\phi \in SENT(\tau_1)$ if $T_1 \models \phi$ then $T_2 \models \exists \bar{x} \Phi^{\#}(\phi)$.*

(iii) We say that T_1 is locally interpretable *in T_2* (with parameters)*, and we write $T_1 \rightarrow_{loc} T_2$, if for every $\phi \in SENT(\tau_1)$ there is a (τ_2, τ_1)–translation scheme Φ with free variables $\bar{x}$ such that if $T_1 \models \phi$ then $T_2 \models \exists \bar{x} \Phi^{\#}(\phi)$.*

The following is clear from the definitions or previous discussions:

Proposition 9.7 *(i) If $T_1 \rightarrow_g T_2$ then $T_1 \rightarrow_{gp} T_2$;*

(ii) If $T_1 \rightarrow_{gp} T_2$ then $T_1 \rightarrow_{loc} T_2$;

(iii) If T_1 is finitely axiomatizable then $T_1 \rightarrow_{gp} T_2$ iff $T_1 \rightarrow_{loc} T_2$;

(iv) There is a finitely axiomatizable theory T_1 and a theory T_2 such that $T_1 \rightarrow_{gp} T_2$ but not $T_1 \rightarrow_g T_2$.

Proof: (i)–(iii) are immediate. (iv) follows from ??. $\square$

We allow here uncountable theories. The interpretability types form a proper class.

9.3 Consistency

Definition 9.8 *Let T be a theory over the vocabulary τ.*

(i) A theory is reflexive *if it contains a sufficiently large fragment of PA and $T \models \phi$ implies $T \models Con(\phi)$.*

(ii) A theory is essentially reflexive *if every extension of T over the same vocabulary is reflexive, or equivalently, if for every τ–sentence ϕ $T \models \phi \rightarrow Con(\phi)$.*

Definition 9.9 *Let T be a theory over the vocabulary τ. T is called a* weak set theory *if there is a τ–formula $\epsilon(x, y)$ such that*

(i) $T \models \exists y \forall x \neg \epsilon(x, y)$

(ii) $T \models \forall x, y \exists z \forall u [\epsilon(u, z) \leftrightarrow (\epsilon(u, x) \wedge u = y)]$

The following is due to Orey [Ore61] and [MPS90].

MPS 3.16

Theorem 9.10 (Orey's Compactness) *Let T_1, T_2 be two theories, T_1 recursively enumerable and T_2 reflexive and $T_1 \rightarrow_{loc} T_2$. Then $T_1 \rightarrow_g T_2$ (even with a unary translation scheme).*

MPS

Theorem 9.11 *Let T_1, T_2 be two theories, T_1 finitely axiomatizable by the formula ϕ and T_2 reflexive. Then $T_1 \rightarrow_{loc} T_2$ iff $T_2 \models Con(\phi)$.*

Smorynski, MPS 3.13

Theorem 9.12 (Feferman,Stern) *(i) Assume $PA \rightarrow_g T$ and T recursively enumrable and consistent. Then $T + Con(T) \not\rightarrow_g T$.*

(ii) There is a sentence ϕ_0 with $PA \models \phi_0$ such that for any T recursively enumrable and consistent $\phi_0 + Con(T) \not\rightarrow_g T$.

MPS 6.6

Theorem 9.13 (Montague) *Let T be a finitely axiomatizable theory with $Q \rightarrow_g T$. Then there is a formula $\phi(x)$ such that for every $n \in \mathbb{N}$*

$$T + \phi(\mathbf{n}) \not\rightarrow_g T + (\forall x(x \neq \mathbf{n} \rightarrow \phi(x))$$

Ryll's theorem Friedman's theorem

9.4 Faithful Interpretations

9.5 Sequential Theories

9.6 Weak Theories

9.6.1 Complete Theories

Mycielski 1977 (Sczcerba)
$ACF_0 \rightarrow_g RCF$ but $RCF \not\rightarrow_g ACF_0$
MPS 7.2

9.7 Interpretability in Finite Models

9.8 Friedman's Theorem

We shall now discuss a beautiful characterization of interpretatibility due to H. Friedman. Friedman proved his theorem in 1976 but never published his own proof. A proof based on Friedman's notes of 1980 was first presented in Smorynski's [Smo85], whose presentation we follow rather closely.

9.9 Friedman's Theorem

The theorem to be presented has two new (with respect to our presentation) ingredients. The first is that instead of considering the consistency assertion

$$T \models CON(T) \rightarrow CON(S)$$

we consider now the weaker statement

$$T_0 \models CON(T) \rightarrow CON(S)$$

where T_0 is a weaker theory than both$T$ and S. H. Friedman introduced such a weak theory EFA (*Elementary Function Arithmetic* which plays a crucial rôle in his theorem. We shall discuss make this more precise below.

The second concerns the complexity of proofs. Instead of considering the statement $CON(T)$ we consider the statement

$$CONTR(T, n, m)$$

which asserts that there exists a proof sequence using T of size m and quantifier complexity n proving a contradiction. Again the choice of the definitions of size and quantifier complexity of a proof is vague here and will be clarified below. If no bound on the size of the proof sequence is required we write

$$CONTRQ(T, n).$$

Now H. Friedman considers finitely axiomatized theories S and T which are at least as strong as EFA and allow the coding of finite sequences of its elements. Such theories are called *adequate*.

At last we define the iterated exponential function as follows:

Definition 9.14 (Iterated Exponential) *The* iterated exponential function $iexp(x, y) : \mathbb{N} \rightarrow \mathbb{N}$ *is defined inductively by*

(i) $iexp(0, y) = 1$ *for every* y*;*

(ii) $iexp(1, y) = y$ *for every* y*;*

(iii) $iexp(x + 1, y) = 2^{iexp(x,y)}$.

We are now in a position to state Friedman's Theorem:

Theorem 9.15 (Friedman 1976) *Let S and T be adequate, finitely axiomatizable theories. Then the following are equivalent:*

(i) $S \rightarrow^{loc} T$*;*

(ii) $S \rightarrow^{gp} T$*;*

(iii) for some constant c

$$EFA \models \forall x(CONTRQ(S,x) \to CONTRQ(T, x+c));$$

(iv) for some constants c, d

$$T \models \forall x, y(CONTR(S,x,y) \to CONTR(T, x+c, iexp(d,y))).$$

9.10 Relative Consistency and Adequate Theories

9.11 Effective Consistency

Let S and T be first order theories. We have seen already that if T is satisfiable and $S \to_{loc} T$ via first order translation scheme then S is satisfiable. Equivalently if S is inconsistent and $S \to_{loc} T$ then T is inconsistent. Note that here both the compactness and the completeness theorem are used.

Now let us look at proof sequences. Fix your favourite proof system for first order logic. The *size* of a formula is the physical size one needs to write down the formula, including parentheses and indices of variables and extra logical symbols. Indices are written in binary. The size of a proof sequence is the sum of the sizes of its formulas including the hypotheses used from S.

If S and T are finitely axiomatizable, $S \to_{gp} T$ via first order translation scheme Φ and

$$\psi_1, \ldots, \psi_n$$

is a proof sequence using hypotheses from S then

$$\Phi^{\#}(\psi_1), \ldots, \Phi^{\#}(\psi_n)$$

is a proof sequence using as only hypotheses from T what is needed to prove $\Phi^{\#}(S)$. Consequently, its

9.12 Variations

10 The Rôle of Presentation

We have seen in the examples ?? that there are several ways of presenting the notion of groups as first order structures, but that the differences matter little, as it is clearly understood how to pass from one presentation to another: the passage is described by first order translation schemes. Many such sitation occur in mathematics. The formalization of Euclidean Plane Geometry as two sorted structure consisting of lines and points with an incidence relation and its counterpart the two–dimensional vector space $\mathbb{R}^2$ may be the most prominent.

In this section we shall discuss this passage between presentation further. In the literature several such discussions exist, with different underlying definitions and focus of interest. However, research in this area has been marginal, and its relevance for the foundation of Computer Science has been widely overlooked.

10.1 Model Engineering

A Database designer is usually given the following task: Model an application domain, say a library, in some data model, say the relational model, such that all relevant transactions can be recorded and efficiently automatized, or at least, interactively processed on computers. A successful completion of such a task has many ingredients, but it will consist of at least the following:

- A choice of the basic relations $r_1, \ldots, r_n$ and its appropriate relation scheme $R = \langle R_1, \ldots, R_n \rangle$.
- A discussion of the criteria leading to the choice of R in terms of independence, redundance, data base normal forms, cf. [GM95] in these Handbooks.
- A discussion of the criteria leading to the choice of R in terms of completeness of the model, i.e. evidence that all the concepts and transactions occuring in the application domain where captured.
- A discussion of the criteria leading to the choice of R in terms query and update languages.
- A discussion of the criteria leading to the choice of R in terms efficiency considerations.

Even if we consider the choice of, say the relational model, justified, two database designer may come up with seemingly radically different solutions. In practice, they may argue passionately about their choices. But very often they may not be aware that their seemingly different solutions could be 'equivalent' for most practical purposes. Moreover, even if they decide to choose different data models (hierarchic, object oriented, entity relationship model), such an underlying equivalence might have been overlooked. If the result of two modelling processes turn out to be equivalent, one should not jump to the conclusion, that one may not have advantages over the other. On the contrary, such advantages may well exist, but they will not be of logical character. They will be uniquely concerned with what computer scientists like to call the user interface. They may be ergonomic, psychological, or connected with traditions. The isolation of such a notion of equivalence of two data models will allow to shift such an argument and address the relevant questions.

10.2 Concepts and their Presentation

Concretely a *concept* is given by a class of structures, the *instances* of the concept. As a class of structures a presentation, its vocabulary (or data model) is chosen for the concept. More abstractely, and independent of its presentation, we can look at the equivalence class of a concept under some notion if *bi–interpretability*. In recursion theory such equivalence classes are called *degrees*. Mycielski, Pudlak and Stern, [MPS90] call such equivalence classes *interpretability types* and a more general concept *chapters (of mathematics)*. We shall call them here *degrees of interpretability*. To define them we have several options, depending on the notion of interpretability we choose. Most of the existing studies of such notions emphasize the proof theoretic aspect. As we also look at logics, whose set of tautologies is not recursively enumerable, we rather stress the semantic approach based on the consequence relation.

10.2.1 Recursive Reducibility of Theories

Theories are countable. The reducibility degrees form a set. We can define a partial order

Proposition 10.1 *There is*

For finitely axiomatizable theories the reducibly degrees form a set. On this set we define a lattice structure L_0.

Theorem 10.2 (A. S. Stern) *L_0 is a distributive lattice.*

Instead we can consider recursive reducibility via functions of low complexity, such as LogSpace or Polynomial Time. Let **C** be any 'reasonably' low leven complexity class.

Problem 10.3 *Study the preorder among finitely axiomatizable theories via* **C***–reducibility.*

10.2.2 Interpretability Types

In the case of recursive reducibility we only required that the function g between formulas be recursive. Now we look at the case where g is of the form $\Phi^{\#}$ for some translation scheme Φ in some logic $\mathcal{L}$.

We allow here uncountable theories. The interpretability types form a proper class.

10.2.3 Chapters of Interpretability

Examples 10.4 (Models of Computation)

- *(Recursive functions and sets) The structure here is $\mathcal{N}$. The set of recursive functions is defined inductively starting with the constant functions and closing under the operations composition, projection and recursion. A set is recursive if its characteristic function is a recursive function.*
- *(Recursive functions on strings)*
- *(Turing acceptable languages)*

Theorem 10.5 (Church's thesis) *All these structures are recursively bi-interpretable*

References

[EFT80] H.D. Ebbinghaus, J. Flum, and W. Thomas. *Mathematical Logic*. Undergraduate Texts in Mathematics. Springer-Verlag, 1980.

[GM95] E. Grädel and J.A. Makowsky. Logical aspects of database theory. In S. Abramsky, D. Gabbay, and T. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 5. Oxford University Press, 1995.

[Mak92] J.A. Makowsky. Model theory and computer science: An appetizer. In S. Abramsky, D. Gabbay, and T. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 1, chapter I.6. Oxford University Press, 1992.

[MPS90] J. Mycielski, P. Pudlák, and A.S. Stern. *A Lattice of Chapters of Mathematics*, volume 426 of *Memoirs of the American Mathematical Society*. Amercan Mathematical Society, 1990.

[Ore61] S. Orey. Relative interpretations. *Zeitschrift für Logic und Grundlagenforschung*, 7:146–153, 1961.

[Rab65] M.A. Rabin. A simple method for undecidability proofs and some applications. In Y. Bar Hillel, editor, *Logic, Methodology and Philosophy of Science II*, Studies in Logic, pages 58–68. North Holland, 1965.

[Smo85] C. Smoryński. Nonstandard models and related developments. In L. Harrington, M. Morley, A. Scedrov, and S. Simpson, editors, *Harvey Friedman's Research on the Foundations of Mathematics*, volume 117 of *Studies in Logic*, pages 179–230. Elsevier Science Publishers, 1985.