Lecture 6

The Tutte polynomials

- The chromatic polynomial

- Tutte polynomial

We follow:

[MGT] B. Bollobás, Modern Graph Theory, Springer 1998

Some notation

Let G be graph $G = \langle V, E, R \rangle$
(with possibly loops and multiple edges)
with $(v, e) \in R$ iff v is a vertex of the edge e.

$G - e$: $V(G - e) = V$, $E(G - e) = E - \{e\}$, i.e. we just delete e.

G/e: For $(v, e) \in R, (v', e) \in R$ we identify (fuse) v and v' and omit the edge e.

e is a **bridge in** G, if $G - e$ has more connected components than G.

e is a **loop in** G, if e has only one vertex incident with it.

B and L denote the graph with a single edge which is bridge or loop respectively. E_n is the graph with n isolated points and no loops.

Vertex colouring, I

Given a graph $G = \langle V, E \rangle$.

A k-vertex colouring is a partition of V into k independent subsets.

The chromatic number $\chi(G)$ is the smallest k, such that a k-vertex colouring exists.

For a positive integer x we denote by $p_G(x)$ the number of x-vertex colourings of G.

For the clique K_n and its complement E_n we have

- $\chi(K_n) = n$, $\chi(E_n) = 1$,
- $p_{K_n}(x) = 0$ for $x \leq n-1$
  $p_{E_n}(x) = x^n$ the number of functions
  from $\{1, \ldots, x\}$ to $\{1, \ldots, n\}$.
- $p_{K_n}(n) = n!$, $p_{E_n}(n) = n^n$.
- $p_{K_n}(n+1) = n$.
  $p_{E_n}(n+1) = p_{E_n}(n) \cdot n = n^{n+1}$

Vertex colorings, II

Given a graph $G = \langle V, E \rangle$ and $e \in E$.

We denote by $G - e$ the graph $\langle V, E - \{e\} \rangle$ and by G/e the graph where the vertices of e are identified and e is dropped (not made into a loop).

Exercise:(MGT, page 150)

$$p_{G-e}(x) = p_G(x) + p_{G/e}(x) \tag{1}$$

$$\chi(G - e) = min\{\chi(G), \chi(G/e)\} \tag{2}$$

Let $G_1, \ldots, G_k$ be the connected components of G. Then

$$\chi(G) = max\{\chi(G_i), i \leq k\} \tag{3}$$

and

$$p_G(x) = \prod_{i=1}^{k} p_{G_i}(x) \tag{4}$$

The chromatic polynomial

Theorem:(MGT: Theorem V.4)

Let H be a graph with $n \geq 1$ vertices, m edges and k connected components. Then

$$p_H(x) = \sum_{i=0}^{n-k} (-1)^i a_i x^{n-i}$$

where $a_0 = 1$, $a_1 = m$ and
$a_i \geq 1$ for $0 \leq i \leq n-k$.

In particular $p_H(x) \in \mathbb{Z}[x]$ is a polynomial

Proof of theorem V.4, I

Induction over $n + m$.

$n + m = 1$ (hence $n = 1$) is trivial
with $p_{E_1}(x) = x$.

If $m = 0$ we have E_n and $p_{E_n}(x) = x^n$.

If $m \geq 1$ we choose an edge $e \in E$.
By induction hypothesis the theorem is true for
$H - e$ and H/e.

We also note that H/e has also k components
and $H - e$ has at least k components.

Proof of theorem V.4, II

Recall that $a_0 = 1, a_1 = m$.

Hence we have for $H - e$

$$p_{H-e}(x) = x^n - (m-1)x^{n-1} + \sum_{i=2}^{n-k} (-1)^i b_i x^{n-i} \qquad (5)$$

with $b_i \geq 0$, and for H/e

$$p_{H/e}(x) = x^{n-1} - \sum_{i=2}^{n-k} (-1)^i c_i x^{n-i} \qquad (6)$$

with $c_i \geq 1$.

If $H - e$ has $k' > k$ many components, $b_i = 0$ for $n - k' < i \leq n - k$.

Proof of theorem V.4, II

Now we use $p_{H-e}(x) = p_H(x) + p_{H/e}(x)$:
and put $a_i = b_i + c_i$

$$p_H(x) = p_{H-e}(x) - p_{H/e}(x) =$$

$$x^n - (m)x^{n-1} + \sum_{i=2}^{n-k} (-1)^i a_i x^{n-i} \qquad (7)$$

where $a_i \geq 1$.

Q.E.D.

Exercise: Let $\pi_r(H)$ denote the number of partitions of $V(H)$ into r independent sets.

$$p_H(x) = \sum_{r=1}^{n} \pi_r \prod_{j=0}^{r-1} (x - j) \qquad (8)$$

Coefficients of $p_H(x)$

Let H be a graph with n vertices and $E(H) = \{e_1, e_2, \ldots e_m\}$ (ordered).

A subset C of $E(H)$ with edge of highest index j is a **broken cycle** if there is $e_l, l \geq j+1$ such that $C \cup \{e_l\}$ is a cycle.

Denote by $a_i(H)$ the number of subsets of $E(H)$ of size i which do not contain a broken cycle.

Theorem:(Theorem V.5.)

$$p_H(x) = \sum_{i=0}^{n-1} (-1)^i a_i(H) x^{n-i} \tag{9}$$

Proof of theorem V.5, I

Lemma: Suppose $e_1 \notin F \subseteq E(H)$.
Then F contains no broken cycle of H iff F contains no broken cycle of $H - e_1$.

Lemma: Suppose $e_1 \in F \subseteq E(H)$. Then F contains no broken cycle of H if(f) $F - \{e_1\} \subseteq E(H/e_1)$ and $F - \{e_1\}$ contains no broken cycle of H/e_1.

Proof of theorem V.5, II

Induction on m:

Again $m = 0$ is trivial. (Check)

Assume that for e_1, both $H - e_1$ and H/e_1 satisfy the theorem.

By abuse of notation we identify $E(H - e)$ and $E(H/e)$ with subsets of $E(H)$. In $E(H/e)$ we give the edge coming from two edges the higher index.

Claim:

$$a_i(H) = a_i(H - e) + a_{i-1}(H/e) \tag{10}$$

This follows from the lemmas.

Q.E.D.

The girth of a graph

Looking at cycles in $E(H)$ one can ask several questions:

- The **girth of** H is the minimal length of a cycle.
- The **circumference of** H is the maximal length of a cycle.

Corollary:(Corollary V.6)
Ley H have n vertices, m edges, girth g and

$$p_H(x) = \sum_{i=o}^{n} (-1)^i a_i x^{n-i}$$

Then $a_i = \binom{m}{i}$ for $i \leq g-2$.
Furthermore, if H has c_g cycles of length g, then $a_{g-1} = \binom{m}{g-1} - c_g$.

Exercises for $p_H(x)$, I

V.45

Show that for a tree T on $n$vertices

$$p_T(x) = x(x-1)^{n-1}$$

V.45

Show that for a forest F on $n$vertices and m edges

$$p_F(x) = x^{n-m}(x-1)^m$$

V.46

Let e be a bridge of H. Show that

$$p_H(x) = \frac{x-1}{x} p_{H-e}(x)$$

e is a bridge if $H-e$ has at least one more connected component than H.

Exercises for $p_H(x)$, II

V.48

Let $G = G_1 \cup G_2$ with $G_1 \cap G_2 = K_r$ the complete graph on r vertices.
Denote by $(x)_r = \pi_r \prod_{j=0}^{r-1}(x - j)$.
Show that

$$p_G(x) = \frac{p_{G-1} \cdot p_{G-2}}{(x)_r} = \frac{p_{G-1} \cdot p_{G-2}}{p_{K-r}(x)}$$

What can you if we replace K_r by some arbitrary H?

Complexity of $p_G(x)$

Given G, how difficult is it to

- compute the coefficients of $p_G(x)$?
- to evaluate $p_G(x)$ for $x = a$?

Exercise:(Folklore) Show that, for $x \geq 3$, computing $p_G(x)$ is $\sharp\mathbb{P}$ hard.

Find a parsimonious reduction of $3COL$ to $3SAT$.

What can we conclude for the above questions?

Given the coefficients, evaluation is fast.

Recall how to compute coefficients of a polynomial $p(x)$ from suffifiently many evaluations.

Rank-generating polynomial $S(G; x, y)$

Given a multigraph (possibly with loops and multiple edges) $G = \langle V(G), E(G) \rangle$, we denote by

- $k(G)$ the number of its connected components.
- The rank $r(G) = \mid V(G) \mid - k(G)$.
- The nullity $n(G) = \mid E(G) \mid - \mid V(G) \mid + k(G)$.
- For $F \subset E(G)$ we denote by $\langle F \rangle$ the spanning subgraph $\langle V(G), F \rangle$.
- $k\langle F \rangle$, $r\langle F \rangle$, $n\langle F \rangle$, denotes the number of components, rank and nullity of the spanning subgraph $\langle F \rangle$.

The **rank-generating polynomial** $S(G; xy) \in \mathbb{Z}[x, y]$ **of** G is defined by

$$S(G : x, y) = \sum_{F \subseteq E(G)} x^{r\langle E \rangle - r\langle F \rangle} y^{n\langle F \rangle}$$

Properties of $S(G : x, y)$

Theorem:(Theorem X.1)

Let G be a multigraph with $e \in E$. Then

$$S(G : x, y) = \begin{cases} (x+1) \cdot S(G-e; x, y) & \text{if } e \text{ is a bridge} \\ (y+1) \cdot S(G-e; x, y) & \text{if } e \text{ is a loop} \\ S(G-e; x, y) + S(G/e; x, y) & \text{if } e \text{ is neither} \end{cases}$$

Furthermore $S(E_n, x, y) = 1$.

The **proof** is in [MGT, Pages 337-339].

Tutte Polynomials, I

The Tutte polynomial is defined via the rank-generating polynomial by

$$T(G;x,y) = S(G;x-1,y-1) = \sum_{F \subseteq E} (x-1)^{r\langle E\rangle - r\langle F\rangle}(y-1)^{n\langle F\rangle}$$

and satisfies

$$T(G:x,y) = \begin{cases} x \cdot T(G-e;x,y) & \text{if } e \text{ is a bridge} \\ y \cdot T(G-e;x,y) & \text{if } e \text{ is a loop} \\ T(G-e;x,y) + T(G/e;x,y) & \text{if } e \text{ is neither} \end{cases}$$

and $T(E_n,x,y) = 1$.

Exercise: Show that for $G = G_1 \cup G_2$ with $G_1 \cap G_2 = \emptyset$ or $G_1 \cap G_2 = E_1$ we have

$$S(G;x,y) = S(G_1;x,y) \cdot S(G_2;x,y)$$

and

$$T(G;x,y) = T(G_1;x,y) \cdot T(G_2;x,y)$$