RC 21813 (98165) 31 July 2000 **Mathematics**

IBM Research Report

The Interlace Polynomial: A New Graph Polynomial

Richard Arratia
University of Southern California

Béla Bollobás
University of Memphis and Trinity College, Cambridge

Gregory B. Sorkin
IBM T.J. Watson Research Center
P. O. Box 218
Yorktown Heights NY 10598

This page intentionally left almost blank.

THE INTERLACE POLYNOMIAL: A NEW GRAPH POLYNOMIAL

RICHARD ARRATIA, BÉLA BOLLOBÁS†, AND GREGORY B. SORKIN

ABSTRACT. We define a new graph polynomial, the interlace polynomial, for any undirected graph. One of our main results is that the polynomial, specified by an intricate recursion relation, is well-defined; in the absence of a direct interpretation of the polynomial, this remains mysterious. The interlace polynomial is not a special case of the Tutte polynomial, nor do we know of any other graph polynomial to which it can be reduced.

For 2-in, 2-out directed graphs D, any Euler circuit induces an undirected "interlace" graph H. In this setting, the interlace polynomial $q(H)$ is equal to the Martin polynomial $m(D)$, a variant of the circuit partition polynomial. There is another connection, between the Kauffman brackets of a link diagram, the Martin polynomial, and in turn the interlace polynomial.

We explore other properties of the interlace polynomial, such as its relations with the component number and independence number of a graph, its extremal values, and its values for various classes of graphs.

A pair of intriguing conjectures remain unproved. More importantly, the fundamental "meaning" of the polynomial is not yet understood.

CONTENTS

1. INTRODUCTION

This work was originally motivated by a problem relating to DNA sequencing by hybridization [ABCS]. At the mathematical heart of the problem was to count the number of 2-in, 2-out digraphs — Eulerian, directed graphs in which vertex has in-degree two and out-degree two — having a given number of Euler circuits.

Date: 27 July 2000.

Key words and phrases. pairing; interlace graph; circle graph; Euler circuit; circuit partition; matrix-tree theorem; BEST theorem; Martin polynomial; Tutte polynomial; Kauffman bracket; extremal.

† Bollobás was supported by NSF grant DMS-9971788.

This may be thought of as the inverse of a standard problem, counting the Euler circuits in a directed graph. That can be done (in polynomial time) with a combination of the matrix-tree theorem [Kir47, Tut48] (see also [Bol98, p. 58]) and the so-called BEST theorem [ST41, dBvAE51] (see also [Bol98, p. 18]). However, this approach does not give the structural information that was needed to solve the inverse problem.

The successful approach led to the graph polynomial discussed here. In addition to counting Euler circuits in a 2-in, 2-out digraph, the interlace polynomial (like the Martin polynomial) can count, for any k, the number of k-component circuit partitions of the graph. In this paper we define the interlace polynomial; treat its connection to the Martin polynomial and the Kauffman brackets of a link diagram; and look into other apparently unrelated properties of the polynomial, some of them only conjectural. Basic questions about the polynomial remain open.

Our approach to counting circuits is based on "interlacings".

Definition 1. *Given an Euler circuit C of a 2-in, 2-out directed graph D, two vertices a and b of D are* interlaced *if C visits them in the sequence $\dots a \dots b \dots a \dots b \dots$. The* interlace graph *$H = H(C)$ corresponding to C has the same vertex set as D, with an edge ab in H if a and b are interlaced in C.*

In the literature, interlace graphs (for example, [RR78]) are also called *circle graphs*, as in [Gol80, dF84, GSH89, Spi94]. Trees constitute a particularly tractable set of circle graphs.

Definition 2. *Given an Euler circuit C with a and b interlaced, a* transposition *on the pair ab is the circuit C^{ab} resulting from exchanging one of the edge sequences from a to b with the other.*

See Figure 1 for an example. Note that if C traverses edge e_1 into a and e_2 out on one visit, and e_3 in and e_4 out on the other visit, then C^{ab} reverses this "resolution", following e_1 with e_4, and e_2 with e_3.

The result stated below as Lemma 4 is due to Ukkonen and Pevzner [Ukk92, Pev95]. Their result is set in a more general context, allowing "rotations" and "3-way repeats"; the relation between their context and 2-in, 2-out digraphs is described in detail in Section 1.4 of [ABCS]. In the context of 2-in, 2-out digraphs, we provide a simple proof of Lemma 4, starting with the case of a circuit having no interlaced pair.

Lemma 3. *If a 2-in, 2-out digraph D has an Euler circuit C with no interlaced pair, then D has only one Euler circuit.*

Proof. Use induction on the order n of D. For $n = 1$ the only case is D consisting of two directed loops on a common vertex, for which there is a single Euler circuit. For $n > 1$ the given circuit C visits the vertices in a sequence $s_1 s_2 \cdots s_{2n}$ in which some vertex a must appear twice consecutively. (The two occurrences of each vertex can be viewed as a pair of matched parentheses, and the absence of interlacings is equivalent to the parentheses being balanced. A vertex a we seek corresponds to an innermost pair of parentheses.) At a there is a loop; contracting out the loop, and then the vertex a, gives a graph D' with the same number of Euler circuits as D. D' also has no interlaced pair, and is of order $n - 1$, so by induction D' has only one Euler circuit, and thus so does D. □

Lemma 4. *For a 2-in, 2-out digraph D, all Euler circuits form a single orbit under transpositions on interlaced pairs.*

Proof. Induction on the order n of D. The unique graph D for the base case $n = 1$ has only one Euler circuit, so the assertion is trivial. Otherwise, let two circuits C, C' of D be given. If there is a vertex a which the two circuits resolve alike — each of a's two in-edges is followed by the same out-edge in C' as it is in C — then use this resolution to contract a out of D, C, and C'. In the contracted case, by induction, one circuit can be converted to the other by a sequence of transpositions, and this sequence lifts to the original case.

If there is no vertex a resolved alike in C and C', then C and C' are different, so by Lemma 3 D must have had some interlaced pair ab. Transposing C' on ab gives a circuit C'' which resolves vertex a oppositely to C', and therefore the same as C. By the previous case, C can be converted to C'', and in turn to C', by a sequence of transpositions. □

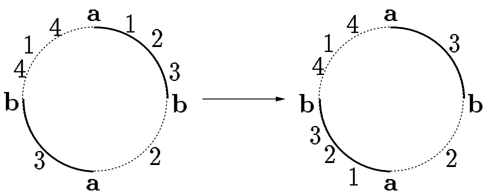


FIGURE 1. In a circuit C, transposing on ab toggles the interlacing relationships of some other pairs of vertices; here 1 and 2, 1 and 3, and 2 and 3. Vertices interlaced with a are interlaced with b after transposition, and vice-versa. Also, the "resolution" at a is reversed by transposition: C follows edge $4a$ with $a1$ and $2a$ with $a3$, while C^{ab} follows $4a$ with $a3$ and $2a$ with $a1$.

Lemma 3 says that if $H(C)$ has no edges, then the Euler circuit is unique. Also, if C has only one interlaced pair ab, transposing on gives a circuit C' in which ab is still the sole interlaced pair, so these two circuits constitute the full orbit; that is, if $H(C)$ has one edge, then there are exactly two Euler circuits. Such observations, studied in [ABCS], suggest a question: does the interlace graph determine the number of Euler circuits? We shall show that it does.

The body of the paper is structured as follows. Section 2 defines a pivoting function on all undirected graphs; the pivot's essential property is that for $H = H(C)$, the graph pivot and the circuit transposition are consistent, as specified by Lemma 7.

In Section 3, this consistency is exploited to show that if D is a 2-in, 2-out digraph, C any of its Euler circuits, and $H = H(C)$ the corresponding interlace graph, then a certain function $q_1(H)$ counts the Euler circuits of D.

Section 4 contains an essential identity for the graph pivot operator.

Section 5 is the central core of the paper. First, it uses the graph-pivot identity of the previous section to prove that a graph *polynomial* q is well-defined on *any* undirected graph G. The polynomial q generalizes the Euler circuit-counting function q_1: $q_G(x)$ evaluated at $x = 1$ is $q_1(G)$.[1] Section 5.2 presents some simple, fundamental properties of the interlace polynomial.

Section 6 shows that when q is applied to an interlace graph H of an Euler circuit C of a 2-in, 2-out digraph D, the evaluations of $q(H)$ at $0, \ldots, k$ can be used to count the partitions of the edges of D into k circuits; this relates the interlace polynomial $q(H)$ to the Martin polynomial $m(D)$.

Section 7 connects the Kauffman brackets of a link diagram to circuit counting, and thus to the Martin and interlace polynomials.

Section 8 computes $q(G)$ for some natural graphs, and some general observations about the interlace polynomial are drawn in Section 9.

"Substituted" and "rotated" graphs are considered in Section 10; calculating $q(G)$ on such graphs is of mild interest in itself, and is essential machinery for the following section.

Section 11 computes extremal properties of the graph polynomial, such the maximum of $q_G(1)$ over all graphs G of order n.

Section 12 lists some open problems. Foremost among these is a "philosophical" question that can be stated right away. We show that $q(G)$ is *defined* for any graph G, and we show that *when* G is an interlace graph of an Euler circuit C of a 2-in, 2-out digraph D, $q(G) = m(D)$, and counts circuit partitions of D. But what is the significance of $q(G)$ generally?

2. THE PIVOT OPERATOR ON GRAPHS

We now define a pivot operator $G \mapsto G^{ab}$ on any graph G. It will have the property that if $H = H(C)$, and vertices a and b are interlaced in C, then the circuit transposition $C \mapsto C^{ab}$ and the pivot operator $H \mapsto H^{ab}$ commute, modulo a vertex relabelling. Figure 2 illustrates the following definition.

[1] Also, Section 3 will only show that q_1 is well-defined for interlace graphs (also known as circle graphs); here we show that q (and therefore q_1 also) is well-defined on any graph.

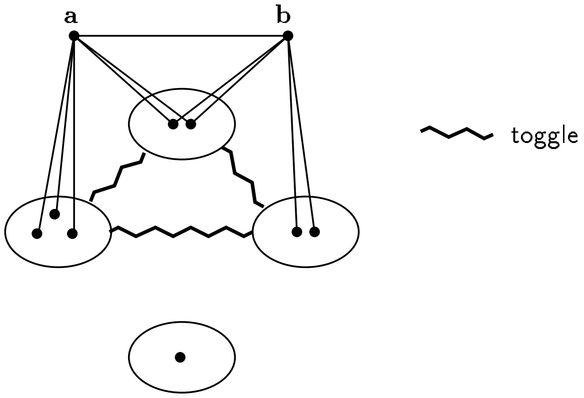


FIGURE 2. Pivoting a graph G on a pair ab (always an edge, in practice). Vertices are divided into 4 classes: those joined to a; to b; to both; or to neither. If vertices x and y lie in different classes among the first three classes named, then if xy was an edge in G, it becomes a non-edge in G^{ab}, and similarly if xy was a non-edge, it becomes an edge.

Definition 5 (Pivot). *Let G be any undirected graph, and ab a pair of distinct vertices in G. Construct the pivot G^{ab} as follows. Partition the vertices other than a and b into four classes:*

1. *vertices adjacent to a alone;*
2. *vertices adjacent to b alone;*
3. *vertices adjacent to both a and b; and*
4. *vertices adjacent to neither a nor b.*

Begin by setting $G^{ab} = G$. For any vertex pair xy where x is in one of the classes (1–3) and y is in a different class (1–3), "toggle" the pair xy: if it was an edge, make it a non-edge, and if it was a non-edge, make it an edge.

Note that $G^{ab} = G^{ba}$. Although the pivot operation is defined for any vertex pair ab, we shall only ever pivot about an *edge* ab.

We will need one other, trivial, operation.

Definition 6 (label-swapping). *For any directed or undirected labelled graph G (or circuit C), and any pair of distinct vertices a and b, let G_{ab} (or C_{ab}) denote the same object with the labels of a and b swapped.*

The pivot function defined above satisfies the following lemma.

Lemma 7. *If $H = H(C)$, and H has an edge ab (i.e., a and b are interlaced in C), then $H^{ab} = (H(C^{ab}))_{ab}$.*

Proof. A transposition on an Euler circuit C may be understood through Figure 1. Vertices are written around the circumference of a circle in the order they are visited by C, and vertices a and b are interlaced if the chord joining the two occurrences of a intersects the chord joining the two occurrences of b.

First, note that after transposition, a and b remain interlaced. Next, look at single vertices other than a and b. A vertex like 4, interlaced with neither a nor b before transposition, remains interlaced with neither. A vertex like 3, interlaced with both a and b, remains interlaced with both. A vertex like 1, interlaced with just a in C, becomes interlaced with just b in C^{ab}, and therefore in $(C^{ab})_{ab}$ is interlaced with just a, as it was in C. After label-swapping, then, incidences between a and b and other vertices are unaffected by transposition.[2]

[2]Since the Lemma's statement $H^{ab} = (H(C^{ab}))_{ab}$ is equivalent to $(H^{ab})_{ab} = H(C^{ab})$, we could have defined the pivot to incorporate the label-swapping, leading to a simpler statement of the Lemma, at the expense of a more complicated pivot operation, and of sacrificing some symmetry in the recursion relation given by eq (2).

Finally, consider incidences between pairs of vertices other than a and b. In the example of Figure 1, 4 remains uninterlaced with 1 and 3, but 1 and 3 become interlaced, matching Definition 5. All other cases can be worked out with similar figures, and all accord with Definition 5; this completes the lemma's proof. □

3. Counting circuits in 2-in, 2-out digraphs

Definition 8. *For any directed graph D, let $r_1(D)$ be the number of Euler circuits of D, and, more generally, let $r_k(D)$ be the number of partitions of the edges of D into k circuits.*

The fact that all Euler circuits of a 2-in, 2-out digraph can be generated through transpositions starting with any Euler circuit C, and transpositions are mirrored by pivots of the interlace graph, will mean that the number of Euler circuits of D can be computed from the interlace graph of any Euler circuit C of D. This is the basis of the following theorem.

Theorem 9. *There exists a function q_1, from the set of circle graphs to the integers, such that for any 2-in, 2-out digraph D with Euler circuit C, the number of Euler circuits of D is equal to $q_1(H(C))$. Moreover, q_1 is uniquely defined by the following recursion:*

$$q_1(H) = \begin{cases} q_1(H-a) + q_1(H^{ab}-b) & \text{if } ab \text{ is an edge of } H \\ 1 & \text{if } H \text{ has no edges.} \end{cases}$$

Proof. In the graph D, vertex a has two in-edges, e_1 and e_2, and two out-edges, e_3 and e_4. (Since we allow loops, the in-edges are not necessarily distinct from the out-edges.) The set of all Euler circuits in D can be partitioned into those that follow e_1 with e_3 (and therefore follow e_2 with e_4), and those that follow e_1 with e_4 (and therefore follow e_2 with e_3). (We do not presume that either class is non-empty.)

It follows that $r_1(D) = r_1(D') + r_1(D'')$. Here D' is derived from D by merging edge e_1 with e_3 (replacing the two with a single new edge going from the tail of e_1 to the head of e_3), merging e_2 with e_4, and deleting the now-isolated vertex a; and D'' is similarly formed by merging e_1 with e_4, merging e_2 with e_3, and deleting a.

Define an analogous deletion operator on Euler circuits, so that $C - a$ means merging the edge preceding and the edge following each occurrence of a, and removing a. Note that an Euler circuit C contains more information than its parent graph D, and in particular $D = D(C)$ is determined by C. In the present case, we have $D = D(C)$; letting $C' = C - a$ and $C'' = C^{ab} - a$, we also have $D' = D(C')$ and (because the transposition C^{ab} switches the allegiances of the two edges into a with the two out) $D'' = D(C'')$. Defining $r_1(C) = r_1(D(C))$, it is immediate that $r_1(C) = r_1(C-a) + r_1(C^{ab} - a)$.

The interlace graphs of these circuits are given by $H' = H(C-a) = H(C) - a$, and (by Lemma 7) $H'' = H(C^{ab} - a) = (H(C))^{ab} - b$. (In the last expression b rather than a is deleted because of label-swapping; see Lemma 7.) Since $C - a$ and $C^{ab} - a$ each have one vertex less than C, suppose inductively that their Euler circuits are counted as per the Theorem, that is, $r_1(D') = q_1(H')$ and $r_1(D'') = q_1(H'')$. Then

$$\begin{aligned} r_1(D) &= r_1(D') + r_1(D'') \\ &= q_1(H-a) + q_1(H^{ab} - b) \\ &= q_1(H), \end{aligned} \tag{1}$$

the last equality following from the recursive definition of $q_1(H)$. This shows both the independence of the quantity $q_1(H)$ from the first pivot ab used in its recursive calculation, and the desired equality $r_1(D) = q_1(H)$. □

The key point of this lies in equation (1), showing the connection between two in–out "resolutions" at a vertex a and the interlace graph pivoted and not pivoted on an edge ab.

4. Pivoting about an edge

Consider the pivoting operation of Definition 5, and recall that $G^{ab} = G^{ba}$. Also note that pivoting is an involution: for any pair ab, $G^{(ab)(ab)} = G$, where the notation $G^{(ab)(cd)}$ represents an iterated pivot $(G^{ab})^{cd}$. We shall need the following properties of pivoting, which are considerably more subtle.

Lemma 10. *Let a, b, and c be distinct vertices of a graph G with $ab, ac \in E(G)$. Then*
(i) $G^{(ab)(ac)(ab)} = G_{bc}$, *and*
(ii) $G^{(ab)(ac)} = (G^{ac})_{bc}$.

Two remarks before the proof. The identities need not hold unless ab and ac are both edges. And, the identities are needed to prove Lemma 14, which in turn is needed for Theorem 12.

Proof. Focus first on identity (i), *in the case where $bc \notin E(G)$*. Proof of (i) consists of checking three types of edges.

First, inspection of Definition 5 shows that the presence or absence in $G^{(ab)(ac)(bc)}$ of any edge in $\{a,b,c\} \times \{a,b,c\}$ is determined by $G|_{\{a,b,c\}}$, the subgraph of G induced by a, b, and c. Checking (i) here merely requires carrying out the pivot operations on the three-vertex graph with edges ab and ac.[3]

Second, for any vertex $u \notin \{a,b,c\}$, the presence or absence in $G^{(ab)(ac)(bc)}$ of any edge in $\{u\} \times \{a,b,c\}$ is determined by $G\,|_{\{a,b,c,u\}}$. Checking (i) here requires checking 8 cases, one for each setting (present/absent) of the three edges ua, ub, and uc. They may be checked "in parallel" by computing $G^{(ab)(ac)(bc)}$ for a single graph G with vertices a, b, c, and 8 more vertices belonging to different equivalence classes as defined by adjacency to a, b, and c. Let us define G not to include any edges except those incident on $\{a,b,c\}$, as such edges are irrelevant to this calculation.

Finally, for any two vertices $u, v \notin \{a,b,c\}$, the number of times the presence/absence of the edge uv is toggled depends on the number of times u and v lie in different classes (1–3) of Definition 5. But the class of u (respectively v) is determined by $G\,|_{\{a,b,c,u\}}$. Note then that the number of times uv is toggled is independent of the initial presence or absence of uv or any other edge not incident on $\{a,b,c\}$. Thus here too it suffices to compute $G^{(ab)(ac)(bc)}$ for the same 11-vertex graph G as above.

In short, in the case $bc \notin E(G)$, identity (i) can be verified by computing $G^{(ab)(ac)(bc)}$ for a single 11-vertex graph with vertices a, b, and c, and 8 other vertices representing the 8 equivalence classes defined by adjacency to a, b, and c.

In the case $bc \in E(G)$ the identity is checked the same way, using the same 11-vertex graph as above but with the edge bc added.

Identity (ii) could be proved using the same pair of graphs, but in fact, (i) and (ii) are equivalent. Since $G \mapsto G^{ab}$ is an involution, $G^{(ab)(ac)(ab)} = G_{bc}$ iff $G^{(ab)(ac)} = (G_{bc})^{ab}$. But $(G_{bc})^{ab} = (G^{ac})_{bc}$, so we are done. □

Remark 11. *If G is connected, then for any edge ab of G, G^{ab} is connected.*

Proof. Edge relations change only between vertices "distinguished" by a and b, but in G^{ab} these vertices remain joined to a or b (or both), and a and b remain joined, so G^{ab} is connected. □

5. A new graph polynomial

We will now show that the integer $q_1(H)$ defined by Theorem 9 can be generalized to a one-variable graph *polynomial* $q(G)(x)$ which is defined for *all* graphs G, not just circle graphs, and which for circle graphs H satisfies $q(H)(1) = q_1(H)$.

We shall call $q(G)$ the *interlace polynomial* of G, and shall write $q(G;x)$, $q(G)(x)$, and $q_G(x)$ interchangeably. Later we will show that when H is an interlace graph of D, $q_H(x)$ can be used to count partitions of D into circuits.

[3] This and the subsequent checks described can certainly be carried out by hand, but we did it by computer instead, with a small amount of manual spot-checking.

5.1. **Definition and well-definedness.** As usual, let us write $\mathcal{G}$ for the class of finite undirected graphs having no loops nor multiple edges.

Theorem 12 (Interlace polynomial). *There is a unique map $q : \mathcal{G} \to \mathbb{Z}[x]$, $G \mapsto q(G)$, such that the following two conditions hold.*

(i) *If G contains an edge ab then*

$$q(G) = q(G-a) + q\left(G^{ab} - b\right). \tag{2}$$

(ii) *On the empty graph E_n, $q(E_n) = x^n$.*

Two quick remarks before proving the theorem. First, in recursively calculating $q(G)$, it is often convenient to apply another rule: if a is an isolated vertex in G, then $q(G) = xq(G-a)$. If this is not obvious, it follows from Proposition 16.

Second, we note that the interlace polynomial at $x = 1$ extends q_1 of Theorem 9 to all graphs.

Corollary 13. *If G is a circle graph, $q_1(G) = q(G;1)$.*

Now we prove Theorem 12.

Proof. If G is a graph of order n *containing an edge ab* then the pivot-reduction formula (2) reduces the computation of $q(G)$ to the computation of q on two graphs of order $n-1$, namely $G-a$ and $G^{ab}-b$. Repeating this process, we reduce $q(G)$ to a linear combination $q(G) = \sum_{k=1}^{n-1} a_k q(E_k)$, where each a_k is a nonnegative integer. By making use of the boundary conditions $q(E_k) = x^k$, we find $q(G)$. Hence, *if there is a function q* then it is unique.

However, it is not clear that *there is* a function q satisfying (i) and (ii). To prove this, we have to show that the same result arises irrespective of the pivot edges chosen in the recursive reduction of G to empty graphs. The proof will be by induction on the order n of G. At $n = 1$ there is nothing to prove, so assume that $n > 1$. By the inductive hypothesis, $q(G-a)$ and $q(G^{ab}-b)$ are both well defined, so we need only show that $q(G-a)+q(G^{ab}-b)$ is independent of the *first* pivot edge ab chosen. (Note that $q^{ab} \stackrel{\text{def}}{=} q(G-a)+q(G^{ab}-b)$, unlike $G^{ab} = G^{ba}$, depends on the *ordered* pair (a,b), that is, on the *oriented* edge ab.) To restate, we must show that for any two oriented edges ab and $a'b'$, $q(G-a)+q(G^{ab}-b) = q(G-a')+q(G^{a'b'}-b')$.

We consider two cases. The first comes if ab and $a'b'$ lie in two different components of G: say $a, b \in G_1$ and $a', b' \in G_2$, with $G = G_1 \cup G_2$. Pivoting G on ab does not affect the component G_2 at all, since all G_2's vertices fall into Definition 5's case (ii). Thus

$$q(G-a) + q(G^{ab}-b) = q((G_1-a)\cup G_2) + q((G_1^{ab}-b)\cup G_2).$$

By the inductive hypothesis, each of the latter terms is well defined, and we may continue as we like. Pivoting on $a'b'$,

$$= q((G_1-a)\cup(G_2-a')) + q((G_1^{ab}-b)\cup(G_2^{a'b'}-b')).$$

By symmetry, this is clearly equal to pivoting first on $a'b'$ and then on ab; thus $q(G-a)+q(G^{ab}-b) = q(G-a')+q(G^{a'b'}-b')$ as was required. Note that isomorphism of G and G' is our only tool for proving equality of $q(G)$ and $q(G')$, so our method was (and always will be) to apply pivot-reduction to each of $q(G)$ and $q(G')$ until the graphs comprising their sums are in one-to-one correspondence.

The more interesting case comes if ab and $a'b'$ lie in the same component of G. If so, G contains a walk $a, b, \dots, a', b'$. Lemma 14 will show that for any sub-walk v_i, v_{i+1}, v_{i+2}, we have $q^{v_i,v_{i+1}}(G) = q^{v_{i+1},v_{i+2}}(G)$, whether v_{i+2} is distinct from v_i (the usual case), or $v_{i+2} = v_i$ (the case we use to re-orient a terminal edge if necessary). Given this, $q^{ab}(G) = \cdots = q^{a'b'}(G)$ as desired. □

The following Lemma is used to prove Theorem 12.

Lemma 14. *Suppose that, for all vertices a and b of G, $q(G-a)$ and $q(G^{ab}-b)$ are both well defined, and let $q^{ab}(G) \stackrel{\text{def}}{=} q(G-a)+q(G^{ab}-b)$.*

(i) *If ab is an edge of G then*

$$q^{ab}(G) = q^{ba}(G).$$

(ii) *If ab and ac are edges of G then*

$$q^{ab}(G) = q^{ac}(G).$$

Before commencing the proof, let us remark that the identities in Lemma 10, and the identities in this Lemma and its proof, are easy to prove, but obscure. We discovered them through a nice alliance of mathematical intuition and computer calculation.[4]

Proof. Both parts of the proof will be by induction on the order of G.

(i) If $d(a) = 1$ or $d(b) = 1$ (or, for that matter, if $\Gamma(a) \cup \{a\} = \Gamma(b) \cup \{b\}$) then $G^{ab} = G$, so

$$q^{ab}(G) \;=\; q(G-a) + q(G^{ab} - b) \;=\; q^{ba}(G).$$

Hence, we may assume that G contains edges ac and bd, distinct from ab. (In fact we could even require that a, b, c and d are distinct vertices, but we do not need to.) Start with

$$q^{ab}(G) = q(G-a) + q(G^{ab} - b).$$

By the inductive hypothesis, both the latter terms are well defined, and we may apply any sequence of pivot edges we like. Pivot the first term on bd, and the second on ac, which is an edge of G^{ab} as it was of G:

$$\begin{aligned} &= \{q(G-a-b) + q((G-a)^{bd} - d)\} + \\ &\qquad \{q(G^{ab} - b - a) + q((G^{ab} - b)^{ac} - c)\} \\ &= q(G-a-b) + q(G^{bd} - a - d) + \\ &\qquad q(G^{ab} - a - b) + q(G^{(ab)(ac)} - b - c). \end{aligned} \tag{3}$$

Symmetrically, swapping the roles of a and b, and those of c and d,

$$\begin{aligned} q^{ba}(G) = q(G-a-b) &+ q(G^{ac} - b - c) \\ &+ q(G^{ab} - a - b) + q(G^{(ab)(bd)} - a - d). \end{aligned} \tag{4}$$

In these expansions of $q^{ab}(G)$ and $q^{ba}(G)$, the first terms are identical, as are the third terms. Also, by Lemma 10 (ii), the second term of (3) is equal to the fourth of (4), because $G^{bd} - a - d \cong G^{(ab)(bd)} - a - d$, and similarly the fourth term of (3) is equal to the second of (4), so $q^{ab}(G) = q^{ba}(G)$.

(ii) This identity is also an immediate consequence of Lemma 10, and is proved similarly, but here the pivot sequences needed do not respect symmetry.

$$q^{ab}(G) = q(G-a) + q(G^{ab} - b);$$

both these terms are well-defined by induction, and pivoting the second on ac,

$$\begin{aligned} &= q(G-a) + \{q(G^{ab} - b - a) + q((G^{ab} - b)^{ac} - c)\} \\ &= q(G-a) + q(G^{ab} - a - b) + q(G^{(ab)(ac)} - b - c). \end{aligned}$$

---

[4]Historically, we began with the graph function q_1, whose well-definedness — on circle graphs — followed from its construction. Generalizing to a graph *polynomial* q was a leap of faith, but computer experiments quickly convinced us that it too was well-defined, on *arbitrary* graphs — that the sequence of pivot elements chosen was irrelevant. We then reasoned, as in the proof of Theorem 12 above, that we would have to prove the equivalence of any two *first* pivot elements, and that this could be boiled down to the equivalence of q^{ab} to q^{ba}, and that of q^{ab} to q^{ac} (or perhaps to q^{ca}).

Consider proving that $q^{ab}(G) = q^{ac}(G)$. This requires finding further pivots to make the two cases "re-converge". Finding these pivot sequences is tricky (and, because the pivoting operation is so clumsy, fearsome to attempt manually) but the possibilities are finite. We must arrange that pivoting on ab gives the same set of graphs as pivoting on either ac or ca — two possibilities. The ab pivot will presumably have to be followed by a pivot on either ac or ca applied to its second term (as the first term lacks the vertex a). Similarly there are two plausible ways to follow up an initial ac or ca pivot. We instructed a computer program to try all the pivot combinations on some examples, and report which combinations matched up in every case. This ferreting-out of the proper sequence of pivots to apply provided the key to the proof.

And,

$$q^{ac}(G) = q(G-a) + q(G^{ac}-c);$$

pivoting the second term on ba (*not* on ab as symmetry would suggest),

$$\begin{aligned} &= q(G-a) + \{q(G^{ac}-c-b) + q((G^{ac}-c)^{ba}-a)\} \\ &= q(G-a) + q(G^{ac}-b-c) + q(G^{(ac)(ab)}-a-c). \end{aligned}$$

By Lemma 10, $(G^{ab}-a-b)_{bc} = G^{(ac)(ab)}-a-c$ and $G^{(ab)(ac)}-b-c = G^{ac}-b-c$, so $q^{ab}(G) = q^{ac}(G)$. □

While we will not exploit the following mild generalization of Theorem 12, it demonstrates that the structure of polynomial multiplication is not required: the powers x, x^2, ... could be replaced by indeterminates $x_1, x_2, \ldots$.

Theorem 15. *There is a unique map $\tilde{q} : \mathcal{G} \to \mathbb{Z}[x_1, x_2, \ldots]$, the $\mathbb{Z}$-module with basis $x_1, x_2, \ldots$, such that the following two conditions hold.*

(i) *If G contains an edge ab then*

$$\tilde{q}(G) = \tilde{q}(G-a) + \tilde{q}\left(G^{ab}-b\right).$$

(ii) *On the empty graph E_n, $\tilde{q}(E_n) = x_n$.*

Proof. Identical to the proof of Theorem 12. □

On the other hand, polynomial multiplication does have a natural role for the interlace polynomial, as shown by Proposition 16, the first of several properties we now consider.

5.2. **Simple properties.** We should first off remark that $q(G;x)$ does not appear to be a specialization of the Tutte polynomial $T(G;x,y)$. At least, for any tree G of order n the Tutte polynomial is always $T(G) = x^{n-1}$, while the interlace polynomial varies widely over trees.

With $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, and $V_1 \cap V_2 = \emptyset$, let $G = G_1 \cup G_2$ denote the disjoint union of the two graphs, so $G = (V_1 \cup V_2, E_1 \cup E_2)$.

Proposition 16. *For any graphs G_1 and G_2 on disjoint vertex sets, $q(G_1 \cup G_2) = q(G_1)q(G_2)$.*

Proof. If $G_1 \cup G_2 = E_n$ then $q = x^n$ and the proposition is trivial. Otherwise use induction on the number of vertices. Pivots in one component don't affect the other, so if ab is an edge in G_1, then $q(G_1 \cup G_2) = q((G_1-a) \cup G_2) + q((G_1^{ab}-b) \cup G_2) = q(G_1-a)q(G_2) + q(G_1^{ab}-b)q(G_2) = q(G_1)q(G_2)$. □

For the Tutte polynomial, the corresponding equality $T(G_1 \cup G_2) = T(G_1)T(G_2)$ holds even if G_1 and G_2 share a single vertex, but the path P_2 suffices to demonstrate that this generalization does not hold for the interlace polynomial.

Remark 17. *For any graph G of order n, $q_G(2) = 2^n$.*

Proof. If $G = E_n$ the conclusion is immediate. Otherwise it follows from a trivial induction on n: $q(G;2) = q(G-a;2) + q(G^{ab}-b;2) = 2^{n-1} + 2^{n-1} = 2^n$. □

Proposition 18. *For any graph G and any edge ab, $q(G^{ab}) = q(G)$.*

Proof. Pivoting G on ab, $q(G) = q(G-a) + q(G^{ab}-b)$. Pivoting G^{ab} on ba, $q(G^{ab}) = q(G^{ab}-b) + q(G^{(ab)(ba)}-a) = q(G^{ab}-b) + q(G-a)$. □

Theorem 19. *Let G be a graph and let H be any induced subgraph of G. Then $\deg(q(G)) \geq \deg(q(H))$. In particular, $\deg(q(G)) \geq \alpha(G)$ where $\alpha(G)$ is the independence number, i.e., the maximum size of an independent set.*

Proof. For the main result, all one has to show is that $\deg(q(G)) \geq \deg(q(G-a))$. This holds because if a is an isolated vertex, then $q(G) = xq(G-a)$; otherwise, for any edge ab, $q(G) = q(G-a) + q(G^{ab} - b)$, and the non-negativity of all terms implies $q(G) \geq q(G-a)$.

The corollary follows by choosing H to be a maximum independent set in G. □

We do not know if there exist graphs for which $\deg(q(G)) \gg \alpha(G)$.

Remark 20. *If $ab \in E(G)$ and $d(a) = 1$ then $q(G) = q(G-a) + xq(G-a-b)$. In particular this holds if G is a forest, a a leaf, and b its parent.*

Proposition 21. *The degree of the lowest-degree term of $q(G)$ is precisely $k(G)$, the number of components of G.*

Proof. That the constant term of q is 0 whenever $|G| \geq 1$ is trivial by induction. Likewise that $q(T)$ has nonzero linear term, using Proposition 20, and the result for forests follows immediately via Proposition 16. Any connected graph G has some non-cut vertex a, and pivoting on any edge pair ab preserves a linear term. □

Theorem 22. *Let $\mu(G)$ denote size of a maximum matching (maximum set of independent edges) in a graph G, and let* $\deg(q)$ *denote the degree of the polynomial q. If G is a forest with n vertices, then* $\deg(q(G)) = n - \mu(G)$.

Proof. Proof by induction on the number of edges of G, with trivial base case when G has no edges. Otherwise, let a be a leaf, b its parent, and consider $G-a-b$ (which may have isolated vertices). Note that $\mu(G-a-b) = \mu(G) - 1$. (Given any matching S on $G-a-b$, $S \cup \{ab\}$ is a one-larger matching on G, so $\mu(G) \geq \mu(G-a-b) + 1$. On the other hand, given any matching S on G, restricting S to the vertices of $G-a-b$ gives a one-smaller matching, so $\mu(G-a-b) \geq \mu(G) - 1$.) Then

$$\begin{aligned}\deg(q(G)) &= \max\{\deg(q(G-a)), \deg(q(G^{ab}-b))\} \\ &= \max\{\deg(q(G-a)), \deg(xq(G-a-b))\}.\end{aligned}$$

Since $\mu(G-a-b) \leq \mu(G-a) \leq \mu(G)$, $\mu(G-a)$ is either $\mu(G)$ or $\mu(G)-1$, so (by the inductive hypothesis) the first term $\deg(q(G-a))$ is either $(n-1)-\mu(G)$ or $(n-1)-(\mu(G)-1)$. Since $\mu(G-a-b) = \mu(G)-1$ (and again using the inductive hypothesis), the second term is $\deg(xq(G-a-b)) = 1 + ((n-2) - (\mu(G)-1)) = n - \mu(G)$. □

Taken together, Remark 17, Theorem 19, Proposition 21, and Theorem 22 mean that from the interlace polynomial $q(G)$ we can read off the graph's order, its component number, an upper bound on its independence number, and (if G is a forest) its matching number.

6. Circuit decompositions and the Martin and interlace polynomials

In this section we relate the interlace polynomial to the enumeration of circuit partitions of a 2-in, 2-out digraph. Various circuit-counting polynomials were defined by Martin [Mar77], and the topic has been explored in [Mar78, Ver81, Ver83, Ver88, Jae88, EM98].

If $H = H(C)$ is the interlace graph of an Euler circuit C of a 2-in, 2-out digraph D of order n, then by Theorem 9, $q_H(1)$ is the number of Euler circuits of D. Since every pairing of the in- and out-edges at each vertex of D leads to a *circuit partition* of D, there are 2^n such circuit partitions; also, it can be seen immediately from Theorem 12 that $q_H(2) = 2^n$. In fact, the entire polynomial $q_H(x)$ can be interpreted in terms of decompositions of D into circuits.

Let us define a somewhat unusual class of objects. We call a digraph *weakly Eulerian* if it is an edge-disjoint union of oriented circuits and "free loops", that is loops without vertices. Equivalently, a weakly Eulerian digraph is a digraph in which every vertex has the same in-degree as out-degree, together with some free loops. These graphs may have multiple edges (and multiple loops), all of which are distinguishable.

Definition 8, of polynomial $r_k(D)$ counting the partitions of a digraph D into k circuits, extends naturally to weakly Eulerian digraphs. Note that $\sum_k r_k(D) > 0$ if and only if D is weakly Eulerian.

Definition 23. *For any weakly Eulerian digraph D, the directed circuit decomposition polynomial is*

$$r(D)(x) = \sum_k r_k(D)x^k.$$

For example if D consists of 3 loops on a single vertex, $r_1(D) = 2$, $r_2(D) = 3$, and $r_3(D) = 1$. If D consists of m loops on a single vertex, $r_k(D) = s(m, k)$, an unsigned Stirling number of the first kind (see for example [Wil90] and [PTW83]), and $r(D)(x) = x(x+1)\cdots(x+m-1)$.

Note that $r_0(D) = 0$ unless D has no edges, in which case $r(D) = 1$.

It is immediate that if $D_1, \dots, D_k$ are the components of D then $r(D) = \prod_{i=1}^k r(D_i)$. To compute $r(D)$, choose any vertex a of D, and let D_σ be the various resolutions at a (matchings of in-edges to out-edges, followed by contraction of the vertex a); then $r(D) = \sum_\sigma r(D_\sigma)$.

Recall that $q(G)$ is defined for any undirected graph G, and we have just defined the circuit decomposition polynomial $r(D)$ for any weakly Eulerian digraph D. In the event that D is an Eulerian (connected) 2-in, 2-out digraph, and G an associated interlace graph, the following theorem describes an interesting connection between them.[5]

Theorem 24. *Let D be an Eulerian 2-in, 2-out digraph, C any Euler circuit of D, and $H = H(C)$ its interlace graph. Then*

$$r(D; x) = xq(H; 1 + x). \tag{5}$$

Proof. Since C conveys more information than D, let us define $r(C) = r(D)$. First, suppose some a and b are interlaced in C (so ab is an edge in H). Recall that, as in the proof of Theorem 9, C and C^{ab} "resolve" the vertex a in the two different ways. Thus,

$$\begin{aligned} r(D; x) &= r(C; x) \\ &= r(C - a; x) + r(C^{ab} - a; x) \end{aligned}$$

which by induction on the order of D (and of C and H)

$$\begin{aligned} &= xq(H - a; 1 + x) + xq(H^{ab} - b; 1 + x) \\ &= xq(H; 1 + x). \end{aligned}$$

If there is no interlaced pair a, b in C, *i.e.*, if H is the empty graph, then C must have a loop on some vertex, and this loop may either be traversed as a separate circuit, giving $xr(C - a; x)$, or may combine with the rest, giving $r(C - a; x)$; in all, this gives $r(C; x) = (1 + x)r(C - a; x)$, and induction completes the proof. ☐

Corollary 25. *Let D be an Eulerian 2-in, 2-out digraph, C any Euler circuit of D, and $H = H(C)$ its interlace graph. If D has r_k partitions into k circuits, then*

$$r_k = \sum_\ell a_\ell \binom{\ell}{k-1} \qquad \text{and} \qquad a_k = \sum_\ell r_{\ell+1}(-1)^{\ell-k}\binom{\ell}{k}.$$

Proof. The first part is immediate from the preceding theorem and the binomial formula. The second follows from the inclusion-exclusion principle (see for example [PTW83]), in particular, from the fact that the matrix M with entries $m_{ij} = \binom{i}{j}$, i and j ranging from 0 to n, has as its inverse a similar matrix with entries $(m^{-1})_{ij} = (-1)^{i+j}\binom{i}{j}$. ☐

Definition 26. *For any weakly Eulerian directed graph D, the Martin polynomial is*

$$m(D; x) = \frac{1}{x-1} r(D; x - 1).$$

[5]The case of disconnected 2-in, 2-out digraphs is not very interesting. Such a graph consists of vertex-disjoint 2-in, 2-out components, $D = D_1 \cup \cdots \cup D_n$. Each D_i has an Euler circuit C_i, and if we define an interlace graph $H = \bigcup H_i$ to be the union of the components' interlace graphs, then $r(D) = \prod r(D_i)$ and $q(H) = \prod q(H_i)$, and it is immediate from Theorem 24 that $r(D) = xq(H; 1 + x)$.

Note that the Martin polynomial is defined on all weakly Eulerian directed graphs, and the interlace polynomial is defined on all undirected graphs. Inverting (5) gives the following corollary.

Corollary 27. *For a 2-in, 2-out directed graph D and any interlace graph H of D,*

$$q(H;x) = \frac{1}{x-1} r(D;x-1) = m(D;x). \tag{6}$$

Corollary 27 connects the interlace polynomial, when restricted to circle graphs, to the Martin polynomial. We now show another connection to the interlace polynomial.

We note that just from inverting (5) to $q(H;x) = \frac{1}{x-1} r(D;x-1)$, it would not be clear — even for circle graphs — that the interlace polynomial's coefficients are non-negative; however, for all graphs, this fact is instantly apparent from the definition of $q(G)$ in Theorem 12.

Let a be a vertex of a 2-in, 2-out digraph D of order n, with arcs e_1, e_2 entering a (say from u_1 and u_2) and arcs f_1, f_2 leaving a (say to v_1 and v_2). Let D'_a and D''_a be the digraphs obtained from D by resolving a in the two possible ways by uniting an incoming edge with an outgoing edge. For example, D'_a is obtained from $D - a$ by adding to it an arc from u_1 to v_1 and an arc from u_2 to v_2, and D''_a is obtained from $D - a$ by adding an arc from u_1 to v_2 and one from u_2 to v_1. If there is a loop at a, so that $u_1 = v_1 = a$, say, then D'_a consists of a 2-in, 2-out digraph with $n-1$ vertices and a vertexless loop, and in D''_a all arcs incident with a are replaced by a single arc. Note that

$$r(D) = r(D'_a) + r(D''_a). \tag{7}$$

An *anti-circuit* is a circuit whose consecutive arcs have opposite orientations. Thus a cycle (cycle meaning circuit with no repeated vertex) is an anti-circuit if every other vertex has in-degree 0 and out-degree 2, the remaining vertices having in-degree 2 and out-degree 0. Every 2-in, 2-out digraph D has a unique decomposition into anti-circuits; we write $a(D)$ for the number of anti-circuits in this decomposition. Martin [Mar77] made the following connection between anti-circuits and the value of $m(D)$ at -1.

Theorem 28 (Martin). *Let D be a 2-in, 2-out digraph with n vertices. Then*

$$m(D;-1) = r(D;-2) = (-1)^{n+a(D)+1} 2^{a(D)}.$$

Proof. We apply induction on n. For $n = 1$ our graph D consists of two loops on a vertex, so $a(D) = 1$, $r(D;x) = x + x^2$ and so $r(D;-2) = 2 = (-1)^{1+2+1}2^1$, as claimed. Suppose then that $n > 1$ and the theorem holds for smaller values of n. Let D'_a and D''_a be as in (7). If a is contained in two alternating circuits then $a(D'_a) = a(D''_a) = a(D) - 1$, so

$$\begin{aligned} r(D;-2) &= r(D'_a;-2) + r(D''_a;-2) \\ &= (-1)^{n-1+a(D)} 2^{a(D)-1} + (-1)^{n-1+a(D)} 2^{a(D)-1} = (-1)^{n+a(D)+1} 2^{a(D)}. \end{aligned}$$

On the other hand, if a is contained in a single alternating oriented circuit then $\{a(D'_a), a(D''_a)\} = \{a(D), a(D)+1\}$, so

$$r(D;-2) = (-1)^{n-1+a(D)+2} 2^{a(D)+1} + (-1)^{n-1+a(D)+1} 2^{a(D)} = (-1)^{n+a(D)+1} 2^{a(D)},$$

completing the proof. ∎

Corollary 29. *Let H be a circle graph. Then*

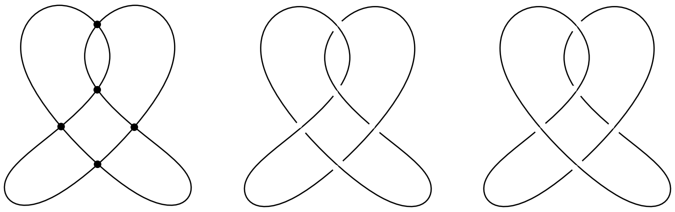
$$q_H(-1) = (-1)^{|H|+1}(-2)^k$$

for some non-negative integer k.

This led us to conjecture in [ABS00] that for all undirected graphs G, not just circle graphs, $q(G;-1) = (-1)^{|G|}(-2)^k$ for some value k; this has recently been proved in [BBCP]. It has implications that may be at least as interesting. If ab is an edge of H and $G = H - a$, then $q(H;-1) = q(G;-1) + q(H^{ab} - b;-1)$. For all three to be powers of 2, it must be that $q(H;-1) \in \{\frac{1}{2}q(G;-1), -q(G;-1), 2q(G;-1)\}$. That is, adding a vertex to any graph G in any way multiplies $q(-1)$ by a factor of $1/2$, -1, or 2; symmetrically, deleting any vertex from any graph H multiplies $q(-1)$ by a factor of $1/2$, -1, or 2.

7. Kauffman brackets and the Martin and interlace polynomials

The aim of this section is to point out some connections between the interlace and Martin polynomials, and some fundamental knot polynomials, namely the Kauffman brackets. Our notation and terminology are standard, and can be found, *e.g.*, in [Bol98, Ch.X., Section 6, pp. 358–378], to which the reader is also referred to for a brief introduction to knots and their polynomials. In particular, the *universe* of a link diagram is the 4-regular *plane* multigraph determined by the strands and crossings of the diagram, where a strand may give several edges. Conversely, every 4-regular plane multigraph with n vertices is the universe of 2^n link diagrams (see Fig. 3). Call an orientation (of the edges) of universe *alternating* if at each vertex



(i) (ii) (iii)

Figure 3. A universe and two of its link diagrams.

the four half-edges alternate, *i.e.*, of any two neighboring half-edges, one is oriented towards the vertex and the other one is oriented away from it. An alternating orientation of a universe gives an *alternating digraph* of the universe. Note that an alternating orientation of a universe G is *not* an orientation of an oriented link diagram with universe G. An alternating digraph of G is a 2-in 2-out digraph, and so is an *oriented* link diagram (not shown) if we ignore over- and undercrossings, but the two digraphs never coincide.

Lemma 30. *Let G be a connected universe with map $M = M(G)$. Then*

1. *the countries of the map M can be 2-colored,*
2. *there is an alternating link diagram whose universe is precisely G,*
3. *the graph G has an alternating orientation.*

Proof. Assertions (i) and (ii) are well known. To see (iii), take a 2-coloring of M and orient the edges in such a way that the boundary of every bounded black face is positively oriented, and the boundary of every bounded white face is negatively oriented, as in Fig. 4. Equivalently, take an alternating link diagram with universe G and orient each edge of G from an overcrossing to an undercrossing. □

In fact, the three assertions in Lemma 30 are trivially equivalent. Also, in each of the three cases above, we have precisely two choices: there are precisely two black-and-white colorings of the countries, there are precisely two alternating diagrams with universe G, and G has precisely two alternating orientations.

Recall that every crossing of a link diagram determines two A-channels and two B-channels, as in Fig. 4 (iv).

Also, a link diagram is alternating if and only if every face of its map has either only A-channels or only$B$-channels (see Fig. 4 (iii)).

Let G be a connected universe, with an alternating digraph D. Let $\mathcal{S} = \mathcal{S}(D)$ be the set of decompositions of G into (oriented) circuits. For $S \in \mathcal{S}$ with $S = \{C_1, \dots, C_k\}$, let $\ell(S)$ be the total number of left turns in the circuits $C_1, \dots, C_k$, let $r(S)$ be the total number of right turns, and let $c(S) = k$ be the number of circuits in S. Note that $\ell(S) + r(S)$ is twice the number of vertices of G, *i.e.*, twice the number of crossings in a link diagram with universe G. Indeed, at each vertex we have either two left turns or two right turns.

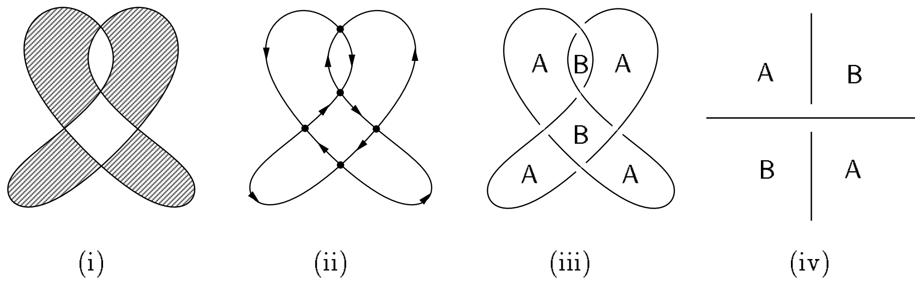


FIGURE 4. (i) A 2-coloring of (the faces of the map of) a link diagram, (ii) the corresponding alternating orientation of its universe, (iii) the associated alternating link diagram, and (iv) the A and B channels.

Define the *square bracket polynomial* of the digraph D as

$$[D](A, B, d) = \sum_{S \in \mathcal{S}(D)} A^{\ell(D)/2} B^{r(D)/2} d^{a(D)-1}.$$

Let us write $[L]$ for the *Kauffman square bracket polynomial* of a link diagram L. The following simple proposition connects the square bracket polynomials with the circuit decomposition polynomial and in turn the Martin and interlace polynomials.

Proposition 31. *Let D be an alternating digraph of a connected universe and let L be the associated alternating link diagram. Then*

$$[D](A, B, d) = [L](A, B, d). \tag{8}$$

Furthermore, writing H for an interlace graph of D, we have

$$[D](1, 1, x) = \frac{1}{x} r_D(x) = m_D(x+1) = q_H(x+1), \tag{9}$$

where r_D is the circuit decomposition polynomial of D and q_H is the interlace polynomial of H.

Proof. Equation (8) is practically a tautology since the Kauffman square bracket $[L]$ is defined by summing over all states S of the universe (or link diagram) the monomials $A^{a_L(S)} B^{b_L(S)} d^{c_L(S)-1}$. If there are n crossings then there are 2^n states, with each state assigning A or B to each crossing u. Each state S corresponds to a decomposition of D into directed circuits by resolving (slicing through) a crossing u along the channel assigned to u by S. There are $a_L(S)$ A-slices, $b_L(S)$ B-slices, and $c_L(S)$ circuits in the resulting decomposition. Thus, to see (8), all we have to note is that in a circuit decomposition given by a state S, at an A-slice we turn left whenever we enter the vertex, and at a B-slice we turn right.

The first equality in (9) is immediate from the definition of the circuit decomposition polynomial, and the second two inequalities are just Corollary 25. □

As Corollary 27 led to a more general property of the interlace polynomial, Proposition 31 suggests a similar (as yet unrealized) possibility.

For $B = A^{-1}$ and $d = -A^2 - A^{-2}$ the Kauffman square bracket becomes the *Kauffman angle bracket*:

$$\langle L \rangle = [L](A, A^{-1}, -A^2 - A^{-2}).$$

This is an *invariant of regular isotopy*, and

$$f[L] = (-A)^{-3s(L)} \langle L \rangle$$

is an *invariant of ambient isotopy*, where $s(L)$ is the self-writhe of L. Applying this with $A = 1$ (so that $B = 1$ and $d = -2$), Proposition 31 implies that $f[L](1) = (-1)^{s(L)}[D](1, 1, -2) = (-1)^{s(L)} q_H(-1)$ is an ambient isotopy invariant of the alternating link diagram L. It is rather unfortunate that, by Theorem 28 or Corollary 29, $|f[L](1)| = |q_H(-1)| = 1$ whenever L is an alternating knot diagram, since if D is the alternating digraph of L then $a(D) = 1$. Hence the invariant $|f[L](1)| = |q_H(-1)|$ cannot be used to prove the weak form of Tait's conjecture that reduced alternating knot diagrams with at least one crossing give non-trivial knots.

8. Interlace polynomials of some simple graphs

Proposition 32. *The interlace polynomials of empty graphs, complete graphs, stars, and complete bipartite graphs are given by*

$$\begin{aligned} q(E_n) &= x^n && \text{for } n \geq 0 \\ q(K_n) &= 2^{n-1}x && \text{for } n \geq 1 \\ q(K_{1n}) &= 2x + x^2 + x^3 + \cdots + x^n && \text{for } n \geq 2 \\ q(K_{mn}) &= (1 + \cdots + x^{m-1})(1 + \cdots + x^{n-1}) + x^m + x^n - 1 && \text{for } m, n \geq 1. \end{aligned}$$

Proposition 33. *For $n \geq 2$, the interlace polynomial of the path P_n (with $n+1$ vertices and n edges) satisfies*

$$q(P_n) = q(P_{n-1}) + xq(P_{n-2}), \tag{10}$$

$$q(P_n) = \sum_{r=0}^{\lfloor n/2 \rfloor} \left\{ \binom{n-r}{r} + \binom{n-r-1}{r} \right\} x^{r+1}, \tag{11}$$

and, for $n \geq 0$ and with $y = \sqrt{1+4x}$,

$$q(P_n)(x) = \frac{(3+y)(y-1)}{4y} \left(\frac{1+y}{2} \right)^{n+1} + \frac{(3-y)(y+1)}{4y} \left(\frac{1-y}{2} \right)^{n+1}. \tag{12}$$

Proof. Formula (10) follows by resolving P_n about an end-edge. We know that $q(P_0) = x$ and $q(P_1) = 2x$, so (11) holds for $n = 0$ and 1. Assuming that (11) holds for $n = 0, 1, \ldots, s$, (10) implies that it holds for $n = s+1$ as well. Equation (12) then follows from

$$\sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n-r}{r} x^r = \frac{1}{y} \left\{ \left(\frac{1+y}{2} \right)^{n+1} + \left(\frac{1-y}{2} \right)^{n+1} \right\}.$$

□

Corollary 34. *For the path P_n, $q(P_n)(1) = F_{n+2}$, the $(n+2)$'nd Fibonacci number (with the convention $F_0 = 0$, $F_1 = 1$).*

Proposition 35. *The interlace polynomials of the cycles C_n are $q(C_3) = 4x$ and, for $n \geq 4$ and with $y = \sqrt{1+4x}$,*

$$\begin{aligned} q(C_n)(x) &= \left(\frac{1-y}{2} \right)^n + \left(\frac{1+y}{2} \right)^n + \frac{y^4 - 10y^2 - 7}{16} && \text{for } n \text{ even,} \\ q(C_n)(x) &= \left(\frac{1-y}{2} \right)^n + \left(\frac{1+y}{2} \right)^n + \frac{y^2 - 5}{4} && \text{for } n \text{ odd.} \end{aligned}$$

Proof. For $n \geq 5$, resolving the cycle C_n at an edge, we find that

$$q(C_n) = q(P_{n-2}) + q(C_{n-2}^+).$$

Write C_s^+ for the graph of order $s+1$ obtained from a cycle C_s by attaching to it a pendant edge. Resolving C_{n-2}^+ at its pendant edge ab, with $d(a) = 1$ and b on the $(n-2)$-cycle, we find that

$$q(C_{n-2}^+) = q(C_{n-2}) + xq(P_{n-4}),$$

and the Theorem follows by induction on n. □

9. Remarks on interlacement and the interlace polynomial

As remarked in the Introduction, we know no direct interpretation of the interlace polynomial, though we suspect that it must have some significance. The connection between the interlace polynomial and circuit decompositions (and the Kauffman bracket), is at a step removed: the interlace polynomial of G tells not about G itself, but rather — and only when G is a circle graph — about circuit decompositions of a graph D having a circuit whose interlace graph is G. Despite this gulf in our understanding, though, we do know quite a bit about the interlace polynomial.

As noted earlier, $q_G(x)$ is not a special case of the Tutte polynomial $T(H)$; for one thing the Tutte polynomial is x^{n-1} for any order-n tree, while q_G varies over trees. We are not aware of any other graph polynomial to which ours is equal nor through which ours can be obtained by transforming the input graph or the output polynomial.

As one would expect, our polynomial does not resolve graph isomorphism. A variety of counterexamples is provided by taking two Euler circuits C_1 and C_2 in a 2-in, 2-out digraph D, and producing their corresponding interlace graphs H_1 and H_2; generally H_1 and H_2 will not be isomorphic, but $q(H_1) = q(H_2)$ since both count circuit partitions of D. Another class of counterexamples is provided by Proposition 16, noting that $q(K_m \cup K_{n-m}) = 2^{n-2}x^2$ regardless of m. Yet another nice counterexample is provided by the 5-cycle, and the 5-cycle with one chord-edge added: both have the polynomial $6x + 5x^2$. In fact, our polynomial does not always even distinguish trees. For example, the path 1–2–3–4–5–6–7 with branches 3–8 and 5–9 has polynomial $2x + 9x^2 + 17x^3 + 13x^4 + 4x^5$, and so does the path 1–2–3–4–5–6 with branches 4–7–8 and 5–9. (There is one other example of order 9, and none smaller, but we expect that there is exponentially small probability that a tree of order n is determined by its interlace polynomial.)

Inspection of a few examples initially suggests that if G and G' are graphs of the same order, with edge sets $E \subseteq E'$, then $q_G(1) \leq q_{G'}(1)$. In fact this is *not* the case. A small example is when G' is the 4-spoke wheel (with edges consisting of the cycle 1, 2, 3, 4 united with the star from 5 to 1, 2, 3, and 4), and G is the same graph with a "circumference edge" deleted (say the edge from 1 to 2). Here $q_G(x) = 6x + 5x^2$ and $q_G(x) = 4x + 4x^2 + x^3$, so $q_G(1) = 11$ and $q_{G'}(1) = 9$.

Instead of drawing an Euler circuit C from a 2-in, 2-out digraph, we could also begin with a circuit, in the form of a "pairing": a list of vertices 1 to n, in which each vertex appears twice, thought of as the order in which a circuit visits these vertices. From a pairing we can construct a unique 2-in, 2-out digraph, and also a unique interlace graph. Since several pairings may lead to the same 2-in, 2-out digraph or interlace graph, the 2-in, 2-out digraphs and the interlace graphs both partition the pairings.

Neither of these partitions *on the pairings* refines the other. In a smallest example, the pairings 1 1 2 2 3 3 and 1 1 2 3 3 2 have isomorphic interlace graphs (both are empty), but non-isomorphic 2-in, 2-out digraphs. Also in a smallest example, the pairings 1 2 3 1 3 4 2 4 and 1 2 4 1 3 4 2 3 have non-isomorphic interlace graphs, but the same 2-in, 2-out digraph.

However, as shown by Propositions 36 and 37, the interlace graphs and 2-in, 2-out graphs partition each other: having pairings with the same interlace graph is an equivalence relation on 2-in, 2-out graphs, and having pairings with the same 2-in, 2-out graph is an equivalence relation on interlace graphs.

Note first that Lemmas 4 and 7 have the following immediate consequence. For a graph G, let $\mathcal{P}(G)$ be the orbit of G under pivoting. Then for every $C \in \mathcal{C}(D)$, $H(\mathcal{C}(D)) = \mathcal{P}(H(C))$.

Proposition 36. *Let $\mathcal{H}(D)$ denote the set of interlace graphs of all Euler circuits of D. Let D and D' be 2-in, 2-out digraphs such that $\mathcal{H}(D) \cap \mathcal{H}(D') \neq \emptyset$. Then $\mathcal{H}(D) = \mathcal{H}(D')$.*

Proof. Choose $C \in \mathcal{C}(D)$ and $C' \in \mathcal{C}(D')$ such that $H(C) = H(C')$. Then $\mathcal{H}(D) = H(\mathcal{C}(D)) = \mathcal{P}(H(C)) = \mathcal{P}(H(C')) = H(\mathcal{C}(D')) = \mathcal{H}(D')$. □

Let $\mathcal{D}(H)$ denote the set of 2-in, 2-out digraphs obtained from all pairings with interlace graph H. Note that for every interlace graph H and edge $ab \in E(H)$, $\mathcal{D}(H) = \mathcal{D}(H^{ab})$. (Any $D \in \mathcal{D}(H)$ is witnessed by a circuit C with $D = D(C)$ and $H = H(C)$; C^{ab} is a witness that $D \in \mathcal{D}(H^{ab})$. Thus $\mathcal{D}(H) \subseteq \mathcal{D}(H^{ab})$; by symmetry, then, $\mathcal{D}(H) = \mathcal{D}(H^{ab})$.) Putting it slightly differently, $\mathcal{D}(H) = \mathcal{D}(\mathcal{P}(H))$.

Proposition 37. *If H and H' are interlace graphs such that $\mathcal{D}(H) \cap \mathcal{D}(H') \neq \emptyset$, then $\mathcal{D}(H) = \mathcal{D}(H')$.*

Proof. Let $D \in \mathcal{D}(H) \cap \mathcal{D}(H')$ and choose $C, C' \in \mathcal{C}(D)$ such that $H(C) = H$ and $H(C') = H'$. $\mathcal{C}(D)$ is a single orbit under transpositions, and contains C and C', so H and H' belong to the same orbit under pivoting: $\mathcal{P}(H) = \mathcal{P}(H')$. Hence $\mathcal{D}(H) = \mathcal{D}(\mathcal{P}(H)) = \mathcal{D}(H')$, as claimed. □

10. Interlace polynomials of substituted and rotated graphs

We now compute interlace polynomials for "substituted" and "rotated" graphs. These are of mild interest in themselves, and are needed for the proofs (and occasionally the statements) of the results in Section 11, on extremal properties of the interlace polynomial.

10.1. **Substituted graphs.** The following definition of graph substitution is recalled from [Bol98].

Definition 38. *If G is a graph with vertices $v_1, \ldots, v_n$, and $G_1, \ldots, G_n$ are arbitrary graphs on disjoint vertex sets, then $G^\star = G[G_1, \ldots, G_n]$ is obtained from $\bigcup_{i=1}^n G_i$ by joining all vertices of G_i to all vertices of G_j whenever $ij \in E(G)$.*

We say that $G^\star$ is obtained from G by *substituting* $G_1, \ldots, G_n$ for the vertices or by *replacing* the vertices with $G_1, \ldots, G_n$. Note that if we replace the vertices of G one by one with the graphs $G_1, \ldots, G_n$, we get the same graph $G^\star$. If each G_i is a complete graph then we call $G^\star$ a *solid graph with template G*, or a *solid G-graph.* Similarly, if each G_i is an empty graph then $G^\star$ is a *thick graph with template G* or a *thick G-graph.*[6]

Clearly, a graph G is a solid graph with a template of order k if and only if $V(G) = V_1 \cup \ldots \cup V_k$ such that each induced subgraph $G|_{V_i}$ is complete and for every pair i, j, $1 \le i < j \le k$, either G contain all $V_i - V_j$ edges, or it contains none of them. Thick graphs have a similar description, except $G|_{V_i}$ is not complete but empty. In particular, the complement $\overline{G}$ of a solid graph G with template H is a thick graph with template $\overline{H}$. Specializing further, the complement of a solid path of length 3 is a thick path of length 3.

Immediately from the definition, substitution is transitive: if $G^\star$ has template G and G has template H, then $G^\star$ has template H. Specifically, we have the following.

Proposition 39. *If $G^\star = G[G_1, \ldots, G_n]$ and $G = H[H_1, \ldots, H_m]$ then there exist graphs $H'_1, \ldots, H'_n$ such that $G^\star = H[H'_1, \ldots, H'_m]$.*

Proof. The result has no mathematical content, but is confusing. Each vertex v_i in H is replaced by a graph H_i to form G. Then each vertex in G — write it as u_i^j to indicate that it is the j'th vertex within graph H_i — is itself substituted with a graph G_i^j. Writing $n_i = |H_i|$, the net result is, specifically,

$$G^\star = H[H_1[G_1^1, \ldots, G_1^{n_1}], \ldots, H_m[G_m^1, \ldots, G_m^{n_m}]].$$

□

In the case of greatest interest, we have the following.

Lemma 40. *If $G^\star$ is a solid G-graph and G is a solid H-graph, then $G^\star$ is a solid H-graph.*

Proof. This is a specialization of the previous proposition in which each H_i is a complete graph and each G_i^j is a complete graph, and thus each $H_i[G_i^1, \ldots, G_i^{n_i}]$ is a complete graph, making $G^\star$ a solid H-graph. □

Lemma 41. *Let a be a vertex of a graph H such that $G = H - a$ is a solid graph with template of order k. Then H is a solid graph with template of order at most $2k+1$.*

Also, if ab is an edge of a solid graph G with template of order k, then G^{ab} is also a solid graph with template of order k.

[6] The case that a substituted graph G_i is the null graph ($|G_i| = 0$) is unnatural but sometimes convenient, for example in the proof of Proposition 56.

Proof. First, let $V_1, \ldots V_k$ be the vertex classes showing that $G = H - a$ is a solid graph with template of order k. For $1 \leq i \leq k$, let $U_i = V_i \cap \Gamma(a)$ and $W_i = V_i - \Gamma(a)$. Then the classes $\{a\}, U_1, W_1, \ldots, U_k, W_k$ (some of which may be empty), show that H is a solid graph with template of order k.

Second, if a and b belong to the same class V_i then $G^{ab} = G$. If, on the other hand, $a \in V_1$ and $b \in V_2$, say, then $V_1' = V_1 \cup \{b\} - \{a\}$, $V_2' = V_2 \cup \{a\} - \{b\}$, $V_3, \ldots, V_k$ show that G^{ab} is a solid graph with template of order k. ◻

Proposition 42. *Let $G^\star = G[K_{m_1}, \ldots, K_{m_n}]$. Then $q(G^\star) = q(G) \cdot 2^{|G^\star|-|G|}$.*

Proof. For any $m_i > 1$, let $H = G[K_{m_1}, \ldots, K_{m_i-1}, \ldots, K_{m_n}]$. Let a, b denote two of the vertices in the K_{m_i}, and pivot-reduce on edge ab. Note that $(G^\star)^{ab} = G^\star$ and that $G^\star - a = (G^\star)^{ab} - b = H$, so that $q(G^\star) = 2q(H)$. The proposition follows inductively. ◻

The simplest case of graph substitution is duplicating a single vertex a, *i.e.*, replacing a with E_2; we will write this $G^\star = G \circ a$. For the case of multiplying the i'th vertex k_i-fold, *i.e.*, taking $G[E_{k_1}, \ldots, E_{k_n}]$, we will write $G^\star = G[k_1, \ldots, k_n]$. As far as the interlace polynomial is concerned, general graph substitution can be expressed in terms of vertex multiplication, via the following theorem.

Proposition 43. *For the general graph substitution $G^\star = G[G_1, \ldots, G_n]$, with $q(G_i) = \sum a(i,k)x^k$, we have*

$$q(G^\star) = \sum_{k_1,\ldots,k_n} a(1,k_1) \cdots a(n,k_n)\, q(G[k_1, \ldots, k_n]).$$

Proof. In $G[G_1, \ldots, G_n]$, the vertices in G_i cannot distinguish among any vertices in G_j, regardless of whether ij is an edge in G. Thus, pivots can be performed within each G_i separately, with no interactions, as in the proof of Proposition 16. ◻

The interlace polynomials $q(G[k_1, \ldots, k_n])$ occurring above can all be expressed as combinations of the the 2^n interlace polynomials $q(H)$, as H runs over the induced subgraphs of G. We carry this out in the next three propositions, first calculating the effect of a single vertex duplication, then calculating the effect of duplicating an arbitrary subset of the vertices, and finally handling the general case.

Proposition 44. *If a is a vertex of a graph G, then*

$$q(G \circ a) = (1+x)q(G) - xq(G-a).$$

Proof. If a is an isolated vertex then $q(G \circ a) = xq(G)$ and $xq(G-a) = q(G)$, proving the assertion. Otherwise, pivot on an edge ab. Consider the "duplicate" vertex a' in the context of the four classes of Definition 5: neighbors $N(a)$ of a alone, neighbors $N(b)$ of b alone, neighbors $N(ab)$ of both, and vertices N_0 that are neighbors of neither. We know that before pivoting, like a: $a' \in N(b)$, a' is adjacent to every vertex in $N(a)$ and $N(ab)$, and a' is adjacent to no vertex $N(b)$ nor N_0. After pivoting, then, $a' \in N(b)$, a' is now adjacent to no vertex in $N(a)$ nor in $N(ab)$, and a' remains adjacent to no vertex in $N(b)$ nor N_0. So in G^{ab}, a' is adjacent only to b, and in $G^{ab} - b$, a' is an isolated vertex: $(G \circ a)^{ab} - b = (G^{ab} - b) \cup E_1$. Thus $q(G \circ a) = q(G \circ a - a) + q((G \circ a)^{ab} - b) = q(G) + xq(G^{ab} - b) = q(G) + x[q(G) - q(G-a)] = (1+x)q(G) - xq(G-a)$. ◻

Proposition 45. *For a graph G with vertices $v_1, \ldots, v_n$, for any $1 \leq m \leq n$, the interlace polynomial for the graph $G^\star$ formed by duplicating $v_1, \ldots, v_m$ can be expressed in terms of the interlace polynomials of the 2^m graphs formed by deleting some subset of $\{v_1, \ldots, v_m\}$, as follows. Let $k_1 = \cdots = k_m = 2$, $k_{m+1} = \cdots = k_n = 1$, and $\ell_{m+1} = \cdots = \ell_n = 1$. Then with $p_1(x) = (1+x)$ and $p_0(x) = -x$,*

$$q(G[k_1, \ldots, k_n]) = \sum_{\ell_1=0}^{1} \cdots \sum_{\ell_m=0}^{1} \left\{ q(G[\ell_1, \ldots, \ell_n]) \prod_{i=1}^{m} p_{\ell_i}(x) \right\}. \tag{13}$$

Proof. This proposition is proved by repeatedly applying Proposition 44. Note first that for $m = 1$, with $v_1 = a$, (13) is precisely Proposition 44. Similarly, for $m = 2$, writing $v_1 = a$, $v_2 = b$, (13) simply says

$$q(G \circ a \circ b) = (1 + x)^2 q(G) - x(1 + x)(q(G - a) + q(G - b)) + x^2 q(G - a - b). \tag{14}$$

Let $H = G \circ a$, so that $H \circ b = G \circ a \circ b$ has $q(H \circ b) = (1 + x)q(H) - xq(H - b)$. Note that $H - b = (G - b) \circ a$ has $q(H - b) = q((G - b) \circ a) = (1 + x)q(G - b) - xq(G - a - b)$. Thus

$$\begin{aligned} q(G \circ a \circ b) &= q(H \circ b) \\ &= (1 + x)[q(H)] - x[q(H - b)] \\ &= (1 + x)[(1 + x)q(G) - xq(G - a)] - x[(1 + x)q(G - b) - xq(G - a - b)], \end{aligned}$$

which shows (14). The general case follows by induction on m, applying Proposition 44 to compute the effect of duplicating the $m + 1$'st, in the graph where the first m vertices have already been duplicated. $\square$

Proposition 46. *Given a graph G on vertices $v_1, \dots, v_n$, with $n \geq 1$, define polynomials $q(k_1, \dots, k_n)(x)$ by*

$$q(k_1, \dots, k_n)(x) = \sum_{\ell_1=0}^{1} \cdots \sum_{\ell_n=0}^{1} (-1)^{n+\ell} q(G[\ell_1, \dots, \ell_n])(x) \prod_{i=1}^{n} \sum_{j=1-\ell_i}^{k_i-1} x^j$$

where $\ell = \sum_{i=1}^{n} \ell_i$. Then for $k_1, \dots, k_n \geq 1$,

$$q(G[k_1, \dots, k_n])(x) = q(k_1, \dots, k_n)(x). \tag{15}$$

Proof. The proof is by induction on $k_1 + \cdots + k_n$, and the basis consists of 2^n cases — those with all $k_i \leq 2$. (The case of (15) with all $k_i = 1$ is trivial, and the other basis cases are given by Proposition 45.) Assume then that some $k_i \geq 3$; without loss of generality we may take $i = 1$. Write $a = v_1$ and $H = G[k_1 - 1, k_2, \dots, k_n]$, so that by Proposition 44, $q(G[k_1, k_2, \dots, k_n])(x) = q(H \circ a) = (1 + x)q(H) - xq(H - a)$. By induction, $q(H)$ and $q(H - a)$ may be expanded as $q(k_1 - 1, \dots, k_n)$ and $q(k_1 - 2, k_2, \dots, k_n)$, in which, for each choice of $\ell_1, \dots, \ell_n$, all factors are equal, except for the factor indexed by $i = 1$. For this factor, the contribution to $(1 + x)q(H) - xq(H - a)$ is $(-1)^{1-\ell_1}$ times

$$(1 + x) \left(\sum_{j=1-\ell_1}^{(k_1-1)-1} x^j \right) - (x) \left(\sum_{j=1-\ell_1}^{(k_1-2)-1} x^j \right) = \left(\sum_{j=1-\ell_1}^{k_1-1} x^j \right),$$

which is the coefficient of $(-1)^{1-\ell_1}$ in the corresponding factor in the expression for $q(k_1, k_2, \dots, k_n)$. This shows $q(G[k_1, k_2, \dots, k_n])(x) = q(k_1, k_2, \dots, k_n)(x)$, completing our proof by induction. $\square$

Corollary 47. *For $r \geq 1$ and $k_1, \dots, k_r \geq 1$, the complete r-partite graph $G^\star = K_r(k_1, \dots, k_r)$ has*

$$q(G^\star)(x) = \frac{x}{2} \prod_{i=1}^{r} (2 + x + \cdots + x^{k_i-1}) + (-1)^r (1 - \frac{x}{2}) \prod_{i=1}^{r} (x + \cdots + x^{k_i-1}). \tag{16}$$

Proof. With $n \equiv r$, this is the special case $G = K_n$ in Proposition 46. In detail, for $l_1, \dots, l_n \in \{0, 1\}$ with $s = l_1 + \cdots + l_n$ note that $G(l_1, \dots, l_n) = K_s$, with $q(K_s)(x) = 2^{s-1}x$ *provided that* $s > 0$, but $q(K_s)(x) = 1 = 2^{s-1}x + (1 - x/2)$ for $s = 0$. Adding this "correction" to the term with $l_1 = \cdots = l_n = 0$

gives a sum which factors:

$$
\begin{aligned}
q(G^\star) &+ (1-\frac{x}{2})\prod_{i=1}^{n}\left(\sum_{j=1}^{k_i-1} x^j\right) \\
&= \sum_{\ell_1=0}^{1}\cdots\sum_{\ell_n=0}^{1}\left\{\frac{x}{2}\, 2^{l_1+\cdots+l_n}\prod_{i=1}^{n}\left[\left(\sum_{j=1-\ell_i}^{k_i-1} x^j\right)(-1)^{1-\ell_i}\right]\right\} \\
&= \frac{x}{2}\prod_{i=1}^{n}\left[2^1\left(1+x+\cdots+x^{k_i-1}\right)-2^0\left(x+\cdots+x^{k_i-1}\right)\right] \\
&= \frac{x}{2}\prod_{i=1}^{r}(2+x+\cdots+x^{k_i-1}).
\end{aligned}
$$

□

10.2. **Rotated graphs.** Let H be a graph including distinct vertices u, v, w, in which uw is an edge and no other edges on w are allowed. Let G be obtained from $H-w$ by toggling the uv relation, so that uv is an edge of G if and only if it is not an edge of H. We say that (G, H) is a *rotation.*

The operation's name comes from the case where uv was an edge in G, and this edge is "rotated" around the pivot point u to become the new edge uw.

Theorem 48. *Let (G, H) be a rotation and $1 \le x$. Then $q_G(x) \le q_H(x)$.*

Proof. We prove the inequality by induction on the order of H. For $|H| = 3$ there are two cases to check, according to whether uv is or is not an edge in H. When uv is an edge, $q_H(x) = 2x + x^2 \ge x^2 = q_G(x)$, as long as $x \ge 0$. When uv is not an edge, $q_H(x) = 2x^2 \ge 2x = q_G(x)$ as long as $x \ge 1$.

In proving the induction step, fix an $x \ge 1$, and assume that $|H| > 3$. We consider the following cases. Other than u and v, G has: (1) only isolated vertices; (2) an edge ab, a and b distinct from u and v; (3) no edge as above, and some vertex c adjacent to v but not u; (4) none of the preceding, and some vertex d adjacent to u but not v; (5) none of the preceding, and some vertex e adjacent to both u and v.

If there is an isolated vertex a of G, then a is also isolated in H, and $q(G; x) = xq(G-a; x) \le xq(H-a; x) = q(H; x)$, where the inequality follows from the inductive hypothesis.

If there is an edge ab independent of u and v, pivot-reduce both G and H on ab. The ab-pivot either toggles the edge uv in both G and in H, or leaves it unchanged in both G and H; also it does not affect the neighborhood of w. Thus (G^{ab}, H^{ab}) is a rotation, and so is $(G^{ab}-b, H^{ab}-b)$. Trivially, $(G-a, H-a)$ is also a rotation, and, by the inductive hypothesis, $q(G; x) = q(G-a; x)+q(G^{ab}-b; x) \le q(H-a; x)+q(H^{ab}-b; x) = q(H; x)$.

Failing the above, it must be that every other vertex in G is adjacent to u or v or both, but to no other vertices. Suppose there is a vertex c adjacent only to v, so $G^{cv} = G$. Then $q(G; x) = q(G-c; x)+q(G-v; x)$. By the inductive hypothesis, $q(G-c; x) \le q(H-c; x)$. Also, $q(H-v; x) = q(H-v-w; x)+q(H-v-u; x) \ge q(H-v-w; x) = q(G-v; x)$. Summing the two inequalities, $q(G; x) \le q(H; x)$.

The next case is that there is a vertex d adjacent only to u. Then $q(G; x) = q(G-d; x) + q(G-u; x) \le q(H-d; x) + q(H-u; x) = q(H; x)$: $q(G-d; x) \le q(H-d; x)$ is by the inductive hypothesis, while $q(H-u; x) = xq(H-u-w; x) = xq(G-u; x) \ge q(G-u; x)$ because $H-u$ and $G-u$ differ just in the isolated vertex w.

The final case is that every vertex in G, other than u and v, is adjacent to both u and v. Pick one such vertex e and pivot-reduce G and H on ev. By the inductive hypothesis, $q(G-e; x) \le q(H-e; x)$. On the other hand, H^{ev} consists of a star on v, an edge uw, and possibly an edge uv; G^{ev} is the same but without the vertex w. Pivot-reducing H^{ev} on wu gives $q(H^{ev}) = q(H^{ev}-w)+q(H^{ev}-u) = q(G^{ev})+xq(G^{ev}-u) \ge q(G^{ev})$. Summing the two inequalities, here too $q(G; x) \le q(H; x)$, and we are done. □

11. Extremal properties of the interlace polynomial

In this section we provide smallest and largest possible values for some properties of the interlace polynomial — such as the value $q(1)$, the degree, and the number of nonzero terms — in terms of the number of vertices $|G|$ of G (its order) and the number of edges $e(G)$ (its size).

We begin with the degree of $q(G)$.

Proposition 49. *For any graph G of order n, $\deg(q(G)) \leq n$, with equality only for $G = E_n$. Also, $q_G(1) \geq 1$, with equality only for $G = E_n$.*

Proof. The first assertion follows from Theorem 19: $\deg(q(G)) = n$ means $\alpha(G) = n$, and G must be the empty graph.

The second assertion is proved by induction on the order of G. If $G = E_n$, then $q_G(1) = 1$ and we are done. Otherwise there is an edge ab in G, and $q(G) = q(G - a) + q(G^{ab} - b) \geq 2$ by the inductive hypothesis. □

Proposition 50. *If $q(G)$ is purely linear, then G is a complete graph.*

Proof. By Theorem 19, $\deg(q(G)) = 1$ means $\alpha(G) = 1$, and G is complete. □

Next, the four results in Propositions 51, 52, 54, and 55 provide lower and upper bounds for $q_G(1)$ in terms of G's order and size.

Proposition 51. *Let G be a graph of order n, having no isolated vertices. Then $q_G(1) \geq n$, with equality if and only if G is a star, or $n = 4$ and G consists of two independent edges.*

Proof. The "if" direction is trivial. To prove the "only if", we apply induction on n. For $n \leq 4$ the result is easily checked, so let us assume that $n > 4$ and the result holds for all smaller values of n. If G is disconnected, say $G = G_1 \cup G_2$ with $|G_1| = n_1 \geq 2$ and $|G_2| = n_2 \geq 2$, then, by the inductive hypothesis, $q_G(1) \geq n_1 n_2 > n$. If on the other hand G is connected, then either G is a tree with $n-1$ edges, or $e(G) \geq n$. Hence, by Proposition 54, $q_G(1) \geq n$, with equality iff G is a tree and is an extremal graph of that Theorem. A graph that is tree and is a complete tripartite graph is a star, completing the proof. □

Proposition 52. *For any graph G of order n, $q_G(1) \leq 2^{n-1}$, with equality if and only if $G = K_n$.*

Proof. The proof is by induction on n, with trivial base case $n = 1$. For $n > 1$, G cannot be the empty graph, so choose an edge ab in G. Then $2^n = q(G) = q(G - a) + q(G^{ab} - b) \leq 2^{n-1} + 2^{n-1}$, so both $q(G - a)$ and $q(G^{ab} - b)$ must achieve their extremal values, and by the inductive hypothesis each must be isomorphic K_{n-1}. Since $G^{ab} - b$ is complete, the neighborhood of a in G^{ab} consists of all vertices (a is connected to b by definition). The neighborhoods of a in G and in G^{ab} are always the same by definition of the pivot operator, thus a is connected to every other vertex in G. Since $G - a \cong K_{n-1}$, $G \cong K_n$. □

Proposition 52 has a nice parallel in the case of trees, and it will be needed to prove Proposition 55.

Lemma 53. *For any tree G of order n, $q_G(1) \leq F_{n+1}$, with equality achieved precisely by a path P_{n-1}.*

Proof. One direction is immediate from Corollary 34: if $G = P_{n-1}$ then $q_G(1) = F_{n+1}$. For the other direction we apply induction on n to show that any tree G achieving the bound must be a path. The base case with $n = 1$ is trivial, as is that $G \neq E_n$.

If $n > 1$, then pick any leaf a, with parent b. The pivot reduction on ab gives $q(G) = q(G-a) + q(G-b) = q(G - a) + xq(G - b - a)$. Evaluating at $x = 1$, $q(G;1) = q(G - a;1) + q(G - b - a;1) \leq F_n + F_{n-1}$ by the inductive hypothesis, with equality only if $G - a$ and $G - a - b$ are both paths. For this to be the case, b must be a terminal of the path $G - a$, and therefore G is also a path. Hence the only tree achieving the upper bound is the path. □

For convenience, we subsume complete bipartite graphs and empty graphs into complete tripartite graphs, by allowing one or two of the three vertex classes to be empty.

Proposition 54. *For every graph G, $q_G(1) \geq e(G) + 1$, with equality if and only if G consists of a complete tripartite graph and isolated vertices.*

Proof. Apply induction on the order n of G. For $n \leq 3$ or $e(G) = 0$ the result holds by inspection, so assume that $n \geq 4$, $e(G) > 0$, and the assertion holds for all smaller values of n.

Let $ab \in E(G)$. Then

$$\begin{aligned} q(G;1) &= q(G-a;1) + q(G^{ab}-b;1) \\ &\geq [(e(G)-d(a))+1] + [e(G^{ab}-b)+1] \\ &\geq [(e(G)-d(a))+1] + [(d(a)-1)+1] \\ &= e(G)+1, \end{aligned}$$

where the the first inequality follows from the inductive hypothesis. This proves the inequality and — since both inequalities hold with equality when G consists of a tripartite graph and isolated vertices — also proves the "if" direction for equality.

If equality holds, then $e(G^{ab}-b) = d(a)-1$, thus all edges of $G^{ab}-b$ are incident to a, so all edges of G^{ab} are incident to a or b. Write Γ for the neighborhood in G^{ab}. Letting $V_a = \Gamma(a) \backslash \Gamma(b)$, $V_b = \Gamma(b) \backslash \Gamma(a)$, and $V_{ab} = \Gamma(a) \cap \Gamma(b)$, pivoting G^{ab} on ab to recover G, we find that G is the complete tripartite graph with vertex classes V_a, V_b, and V_{ab}. (Note that V_a and V_b are both non-empty, since $b \in V_a$ and $a \in V_b$.) □

The next extremal proposition, bounding $q_G(1)$ by a Fibonacci number, uses a property of graph rotation from Section 10.2.

Proposition 55. *For any connected graph G with m edges, $q_G(1) \leq F_{m+2}$, with equality achieved precisely by a path P_m. If G consists of components G_i with sizes m_i, then $q_G(1) \leq \prod F_{m_i+2}$. Always, if G has m edges then $q_G(1) \leq 2^m$, with equality if and only if G consists of m independent edges, and isolated vertices.*

Proof. The proposition's essence is its first assertion. One direction is immediate from Corollary 34: if $G = P_m$ then $q_G(1) = F_{m+2}$.

For the other direction, suppose that $q_G(1) \geq F_{m+2}$. If G is a tree, by Lemma 53, it can only achieve the bound $q(G;q) = F_{n+1} = F_{m+2}$ if it is a path. Otherwise, by the explicit calculations in Proposition 35, G cannot be a simple cycle, and so it has some vertex v of degree at least 3. Choose any edge ab in a cycle of G and not incident to v, and "rotate" the edge out; that is, delete it, and add a new vertex incident to a alone. Repeat until no cycles remain. As we only deleted cycle edges, the graph remains connected. By Theorem 48, each rotation can only increase $q(1)$, yet the final tree, whose vertex v shows it not to be a path, has $q(1) < F_{m+2}$. Thus $q_G(1) < F_{m+2}$, completing the proof of the main assertion.

The result for graphs with several components follows immediately by Proposition 16. Also, $F_{m+2} \leq 2^m$, with equality iff $m = 0$ or 1, proving the final statement. □

Proposition 52 allows an extension, given by Proposition 56. It and the following results rely on properties of graph substitution from Section 10.1.

Proposition 56. *For any graph G of order n, the second-maximum value of $q_G(1)$ is $\frac{3}{4}2^{n-1}$, and is achieved exactly when G is a solid P_2-graph but not a complete graph. (That is, G is composed of a pair of complete graphs sharing some but not all their vertices.)*

Proof. Observing that a complete graph is a special case of a solid P_2-graph, and is covered by Proposition 52, the present proposition is equivalent to: $q_G(1) \geq \frac{3}{8}2^n$ precisely when G is a solid P_2-graph (where one of the three components may be of size 0).

With a little work, we can check that the assertion holds for $n \leq 7$, so assume that $n > 7$ and the assertion holds for graphs of orders up to $n-1$. Let G be a graph of order $n > 7$ with $q_G(1) \geq \frac{3}{8}2^n$. Then G is not empty; let $ab \in E(G)$. Since $q(G)(1) = q(G-a)(1) + q(G^{ab}-b)(1)$, either $q(G-a)(1) \geq \frac{3}{8}2^{n-1}$ or $q(G^{ab}-b)(1) \geq \frac{3}{8}2^{n-1}$. But then, by the inductive hypothesis, either $G-a$ or $G^{ab}-b$ is a solid P_2-graph. By the first part of Lemma 41, either G or G^{ab} is a solid graph of order n with a template H of order $h \leq 7$. By Proposition 42, we have $\frac{3}{8}2^n \leq q(H)(1) \cdot 2^{(n-1)-h}$, so $q(H)(1) \geq \frac{3}{8}2^h$ and thus H is a solid P_2-graph. But then so is at least one of G and G^{ab}. In the second case, G too is a solid P_2-graph, by the second part of Lemma 41. (If the template of G were any order-3 graph other than P_2, the template would still have to be connected — since G^{ab} is and therefore G is — so it could only be K_3, which as a template may be regarded as a special case of P_2 two of whose three groups have size 0.) Hence G is a solid P_2-graph, as claimed. ☐

In fact, the proof above gives the following assertion. Suppose that every graph of order $n \leq 2k+1$ with $q_G(1) \geq c2^n$ is a solid G_i-graph for some $i = 1, \dots, \ell$, where $G_1, \dots, G_\ell$ are graphs of order at most k. Let now G be a graph of order n with $q_G(1) \geq c2^n$. Then G is a solid G_i-graph for some $i, 1 \leq i \leq \ell$.

Just as there is a large gap between $q_G(1)$'s maximum value of 2^{n-1} and its second-maximum value of $\frac{3}{4}2^{n-1}$, there seem to be other such gaps. (Table 6 in [ABCS] exhibits some, albeit only for circle graphs.) The assertion above might be used to prove the following conjecture.

Conjecture 57. *There are constants $c_1 = \frac{1}{2} > c_2 > \dots$ such that for every $k \geq 1$ and n sufficiently large, then, first, there are graphs $G_1, \dots, G_k$ of order n such that $q(G_i)(1) = c_i 2^n$, and, second, if G is any graph of order n with $q_G(1) \geq c_k 2^n$, then $q_G(1) = c_i 2^n$ for some $i \leq k$.*

Lemma 58. *Let G be a connected graph of order at least $\ell + 4$ such that every connected induced subgraph of it with $\ell + 3$ vertices is a solid path of length at most ℓ. Then G is a solid path of length at most ℓ.*

Proof. We may clearly assume that $\ell \geq 3$, for some two vertices a and b the graph $G-a-b$ is a solid ℓ-path, and the graphs $G-a$, $G-b$ and $G-a-b$ are connected. Let P be an induced ℓ-path in $G-a-b$. Since the induced subgraph of G with vertex set $V(P) \cup \{a\} \cup \{b\}$ is a solid ℓ-path, so is the entire graph G. ☐

Proposition 59. *Let G be a graph such that $q(G)$ has precisely two non-zero terms. Then one of the components of G is a solid path of length 2 or 3, and all other components are complete graphs.*

Proof. It is easily seen that in proving this we may assume that G is an incomplete connected graph of order $n \geq 6$, and the assertion holds for graphs of smaller order. Then every connected incomplete induced subgraph H of G has precisely two non-zero terms, so H is a solid path of length at most 3. Writing ℓ for the maximal length of an induced path in G, Lemma 58 implies that G itself is a solid path of length ℓ. ☐

Proposition 60. *If $|G| = n \geq 3$ and $q(G)$ has $n-1$ nonzero terms, then $q(G) = 2x + x^2 + \cdots x^{n-1}$, and G is an n-star.*

Proof. Since $2^n = q_G(2) = \sum_{i=1}^n a_i 2^i$ with each a_i a non-negative integer, we can only have $q_G(x) = 2x + x^2 + \cdots x^{n-1}$. By Proposition 21, since $q(G)$ has nonzero linear term, G is connected. Choose an edge ab of G such that $G-a$ is connected. In particular, for $n \geq 3$, $G-a$ is non-empty, so (by Proposition 49) $\deg(q(G-a)) \leq n-2$. Thus the x^{n-1} term of $q(G)$ must belong to $q(G^{ab}-b)$. As $q(G^{ab}-b)(2) = 2^{n-1}$, we can only have $q(G^{ab}-b) = x^{n-1}$, and so (again by Proposition 49) $G^{ab}-b = E_{n-1}$. Since G^{ab} is connected (see Remark 11), it must be the star with center b. We conclude that $G = G^{(ab)(ab)}$ is also the star with center b. ☐

12. Open problems

To paraphrase Oronce Finé's proposed 17th century name for Antarctica,[7] the interlace polynomial is *terra nuper inventa sed non plene examinata* — territory newly discovered but not yet explored. We have presented what we know about the polynomial, but many questions remain. Here, we draw attention to a related pair of conjectures.

For the first conjecture, we have numerical evidence compiled from thousands of random graphs of orders up to 13, as well as all graphs of orders up to 8. To interpret its second assertion, recall from Theorem 24 that when G is a circle graph with an associated digraph D, $xq(G; 1+x) = r(D; x)$ is the circuit counting polynomial of D.

Conjecture 61. *For any graph G, the sequence of coefficients of $q_G(x)$ is unimodal: non-decreasing up to some point, then non-increasing. The associated polynomial $xq_G(1+x)$ likewise has unimodal coefficients.*

[7] According to Sara Wheeler in *Terra Incognita*.

This conjecture is reminiscent of an outstanding pair of conjectures about the chromatic polynomial, namely that its coefficient sequence is always unimodal [Rea68] and in fact log-concave [Hog74] (see [Wel93]). There are two reasons for caution. First, a similar conjecture for the Tutte polynomial was falsified by counterexample in 1993 [Sch93].

Second, the interlace polynomial's coefficients are certainly not log-concave; the smallest counterexample is $q(K_{1,3}) = 2x + x^2 + x^3$. (We know of no case where the coefficients of the associated polynomial $r(x)$ fail to be log-concave.) This means that, unlike all our other statements about the interlace polynomial, it does not suffice to prove Conjecture 61 for connected graphs: while a product of log-concave polynomials is log-concave, a product of unimodal polynomials is not necessarily unimodal. (Even squaring a unimodal polynomial can result in a polynomial with an arbitrarily large number of modes, as shown in [Sat92].)

We have made little headway in proving the conjecture; for instance we have not even shown that coefficient sequences have no internal zeros, *i.e.*, that the nonzero coefficients form a single block. And, let us note that the second part of the conjecture is not a trivial consequence of the first: it is easy to construct a unimodal polynomial $p(x)$ for which $p(1+x)$ is not unimodal, for example $p(x) = 1000x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12}$.

If G is not the interlace graph of any 2-in, 2-out digraph, then $xq_G(1+x)$ has no interpretation as a circuit decomposition polynomial. However, Conjecture 61 would imply that for any 2-in, 2-out digraph D, the number of partitions r_k into k directed circuits is unimodal in k. This leads us to speculate the following.

Conjecture 62. *For any weakly Eulerian graph or digraph, the number of partitions into k circuits is unimodal in k.*

Beyond Conjectures 61 and 62 specifically, there are general questions about circuit counting. For Eulerian graphs and digraphs, are the circuit decomposition and Martin polynomials mere generating functions, or do they have interesting algebraic properties, as the interlace polynomial does? For $k > 1$, is there a polynomial-time method for counting k-circuit partitions of graphs, directed graphs, or even 2-in, 2-out directed graphs?

There are also basic, unanswered questions about the interlace polynomial. Can $q(G)$ be generalized from graphs to matroids? Can it be computed efficiently? An exponential-time computation of $q(G)$ is immediate from the definition, and we suspect that no polynomial-time computation is possible.[8]

How many different polynomials q_G are there, say for graphs of order n? How many graphs may share a single polynomial? Is there a way to recognize a polynomial obtainable as $q(G)$? What can we say about $q(G)$ when G is a random graph?

Again, though, the real question is this: What does $q(G)$ compute, say when G is not a circle graph?

Acknowledgments

We thank David Gamarnik for fruitful discussions.

References

[ABCS] Richard Arratia, Béla Bollobás, Don Coppersmith, and Gregory B. Sorkin, *Euler circuits and DNA sequencing by hybridization*, Discrete Applied Mathematics, to appear.

[ABS00] Richard Arratia, Béla Bollobás, and Gregory B. Sorkin, *The interlace polynomial: A new graph polynomial*, Proceedings of the Eleventh Annual ACM–SIAM Symposium on Discrete Algorithms (San Francisco, CA), January 2000, pp. 237–245.

[BBCP] P.N. Balister, B. Bollobás, J. Cutler, and L. Pebody, *The interlace polynomial of graphs at −1*, unpublished manuscript.

[Bol98] Béla Bollobás, *Modern graph theory*, Graduate Texts in Mathematics, vol. 184, Springer, New York, 1998.

[dBvAE51] N.G. de Bruijn and T. van Aardenne-Ehrenfest, *Circuits and trees in oriented graphs*, Simon Stevin **28** (1951), 203–217.

[8] At least for circle graphs H, $q_H(1)$ can be computed in polynomial time: find a circle arrangement C for H [Spi94] to obtain a 2-in, 2-out graph D, and count its Euler circuits with a combination of the directed matrix-tree theorem [Kir47, Tut48] (see also [Bol98, p. 58]) and the so-called BEST theorem [ST41, dBvAE51] (see also [Bol98, p. 18]). By the same token, if the full circuit-counting polynomial $r(D)$ for a 2-in, 2-out digraph D cannot be computed in polynomial time, this proves that $q(G)$ cannot be computed in polynomial time.

[dF84] Hubert de Fraysseix, *A characterization of circle graphs*, European Journal of Combinatorics **5** (1984), 223–238.

[EM98] Joanna A. Ellis-Monaghan, *New results for the Martin polynomial*, J. Combin. Theory Ser. B **74** (1998), no. 2, 326–352.

[Gol80] Martin C. Golumbic, *Algorithmic graph theory and perfect graphs*, Academic Press, New York, 1980.

[GSH89] Csaba P. Gabor, Kenneth J. Supowit, and Wen-Lian Hsu, *Recognizing circle graphs in polynomial time*, Journal of the ACM **36** (1989), 435–473.

[Hog74] S. Hoggar, *Chromatic polynomials and logarithmic concavity*, Journal of Combinatorial Theory (B) **16** (1974), 248–254.

[Jae88] François Jaeger, *On Tutte polynomials and cycles of plane graphs*, J. Combin. Theory Ser. B **44** (1988), no. 2, 127–146.

[Kir47] G. Kirchoff, *Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird*, Ann. Phys. Chem. **72** (1847), 497–508.

[Mar77] P. Martin, *Enumérations eulériennes dans les multigraphes et invariants de Tutte-Grothendieck*, Ph.D. thesis, Grenoble, 1977.

[Mar78] Pierre Martin, *Remarkable valuation of the dichromatic polynomial of planar multigraphs*, J. Combin. Theory Ser. B **24** (1978), no. 3, 318–324.

[Pev95] Pavel Pevzner, *DNA physical mapping and alternating Eulerian circuits in colored graphs*, Algorithmica **13** (1995), 77–105.

[PTW83] George Pólya, Robert E. Tarjan, and Donald R. Woods, *Notes on introductory combinatorics*, Birkhäuser Boston Inc., Boston, Mass., 1983.

[Rea68] R.C. Read, *An introduction to chromatic polynomials*, Journal of Combinatorial Theory **4** (1968), 52–71.

[RR78] R.C. Read and P. Rosenstiehl, *On the Gauss crossing problem*, Combinatorics, Vol. 2 (Keszthely, Hungary) (A. Hajnal and V.T. Sós, eds.), Colloquia Mathematica Sociatatis János Bolyai 18, North Holland, Amsterdam, 1978, pp. 843–876.

[Sat92] Ken-Ito Sato, *Convolution of unimodal distributions can produce any number of modes*, Annals of Probability **21** (1992), no. 3, 1543–1549.

[Sch93] Werner Schwärzler, *The coefficients of the Tutte polynomial are not unimodal*, J. Combin. Theory Ser. B **58** (1993), no. 2, 240–242.

[Spi94] Jeremy Spinrad, *Recognition of circle graphs*, Journal of Algorithms **16** (1994), 264–282.

[ST41] C.A.B. Smith and W.T. Tutte, *On unicursal paths in a network of degree 4*, American Mathematical Monthly **48** (1941), 233–237.

[Tut48] W.T. Tutte, *The dissection of equilateral triangles into equilateral triangles*, Proc. Cambridge Philos. Soc **44** (1948), 463–482.

[Ukk92] E. Ukkonen, *Approximate string-matching with q-grams and maximal matches*, Theoretical Computer Science **92** (1992), 191–211.

[Ver81] M. Las Vergnas, *Eulerian circuits of 4-valent graphs imbedded in surfaces*, Algebraic Methods in Graph Theory (Szeged, Hungary, 1978), Coll. Math. Soc. J. Bolyai, vol. 25, North-Holland, Amsterdam, 1981, pp. 45–477.

[Ver83] M. Las Vergnas, *Le polynôme de Martin d'un graphe Eulerian*, Annals of Discrete Math. **17** (1983), 397–411.

[Ver88] M. Las Vergnas, *On the evaluation at (3,3) of the tutte polynomial of a graph*, Journal of Combinatorial Theory, Series B **44** (1988), 367–372.

[Wel93] D.J.A. Welsh, *Complexity: Knots, colourings, and counting*, London Mathematical Society Lecture Notes Series, no. 186, Cambridge University Press, Cambridge, England, 1993.

[Wil90] Herbert S. Wilf, *generatingfunctionology*, 2nd ed., Academic Press, San Diego, CA, 1990.

(Richard Arratia) Univ. of Southern California, Department of Mathematics, Los Angeles CA 90089, USA
E-mail address: rarratia@math.usc.edu

(Béla Bollobás) Univ. of Memphis, Department of Mathematical Sciences, Memphis TN 38152, USA, and Trinity College, Cambridge CB2 1TQ, U.K.
E-mail address: bollobas@msci.memphis.edu

(Gregory B. Sorkin) IBM T.J. Watson Research Center, Department of Mathematical Sciences, Yorktown Heights NY 10598, USA
E-mail address: sorkin@watson.ibm.com