P versus NP over Various Structures

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PART 4: Real versions of classical problems

jointly with J.A. Makowsky

Outline today

1. Introduction
2. Transfer results
3. Inside $\mathrm{NP}_{\mathbb{R}}$
4. Recursion theory over $\mathbb{R}$

1. Introduction: Recall basics in BSS complexity theory

BSS model of computability and complexity over $\mathbb{R}$ and $\mathbb{C}$:

Algorithms allow as basic steps arithmetic operations $+, -, \bullet$ as well as test-operation: $\geq$ over $\mathbb{R}$ and $=$ over $\mathbb{C}$

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Decision problem:	$L \subseteq \mathbb{R}^* := \bigsqcup_{n \geq 1} \mathbb{R}^n$
Size of problem instance:	number of reals specifying input
Cost of an algorithm:	number of operations

Important: Algorithms are allowed to introduce finite set of parameters into its calculations: Machine constants

Definition (Complexity class $P_{\mathbb{R}}$)

$L \in P_{\mathbb{R}}$ if efficiently decidable, i.e., number of steps in an algorithm deciding whether input $x \in \mathbb{R}^*$ belongs to L polynomially bounded in (algebraic) size of input x

Example

Solvability of linear system $A \cdot x = b$ by Gaussian elimination;

Existence of real solution of univariate polynomial $f \in \mathbb{R}[x]$

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The running time is polynomially bounded in (algebraic) size of input x (and thus, only polynomially bounded y's are relevant)

Example

1.) Quadratic Polynomial Systems QPS (Hilbert Nullstellensatz):

Input: $n, m \in \mathbb{N}$, real polynomials in n variables

$p_1, \ldots, p_m \in \mathbb{R}[x_1, \ldots, x_n]$ of degree at most 2; each p_i depending on at most 3 variables;

Do the p_i's have a common real root?

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$\mathrm{NP}_{\mathbb{R}}$-verification for solvability of system

$$p_1(x) = 0 \ , \ \ldots, p_m(x) = 0$$

guesses solution $y^* \in \mathbb{R}^n$ and plugs it into all p_i's ; obviously all components of y^* have to be seen

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Main open problem: Is $\mathrm{P}_{\mathbb{R}} = \mathrm{NP}_{\mathbb{R}}$?

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Remark.

Similar definitions for structures like $\mathbb{C}$ (with $=?$ test), groups, vector spaces, ...

Theorem (Blum-Shub-Smale '89)

a) *The Hilbert-Nullstellensatz problem* $QPS_{\mathbb{R}}$ *is* $NP_{\mathbb{R}}$*-complete. Considered as problem* $QPS_{\mathbb{C}}$ *over* $\mathbb{C}$ *it is* $NP_{\mathbb{C}}$*-complete.*

b) *The real Halting problem* $\mathbb{H}_{\mathbb{R}}$ *is undecidable in the BSS model: Given a machine* M *(as codeword in* $\mathbb{R}^*$*) together with input* $x \in \mathbb{R}^*$*, does* M *halt on* x*?*

c) *Other undecidable problems:* $\mathbb{Q}$ *inside* $\mathbb{R}$*, the Mandelbrot set as subset of* $\mathbb{R}^2$

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c) *Other undecidable problems:* $\mathbb{Q}$ *inside* $\mathbb{R}$*, the Mandelbrot set as subset of* $\mathbb{R}^2$

Both $\mathbb{H}_{\mathbb{R}}$ and $\mathbb{Q}$ are semi-decidable, i.e., there is a BSS algorithm that halts precisely on inputs from these sets.

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All problems in $\mathrm{NP}_{\mathbb{R}}$ are decidable in simple exponential time; similarly for $NP_{\mathbb{C}}$.

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Proof.

Difficulty: uncountable search space; requires quantifier elimination algorithms for real/algebraically closed fields

Long history starting with Tarski; fundamental contributions by Collins, Heintz et al., Grigoriev & Vorobjov, Renegar, Basu & Pollack & Roy, ...

Effective Hilbert Nullstellensatz: Giusti & Heintz, Pardo, ...

General guideline for topics treated below:

What about important questions and results in the Turing model when studied in new computational frameworks?

- P versus NP question?
- impact of results in new models for Turing model?
- and vice versa?
- benefit of different (mathematical) methods available for studying computability on a structure, for example, separation of complexity classes?
- impact of using real or complex constants in an algorithm?
- ...

We shall see that all kind of answers occur such as

- (almost) trivial transfer of similar statements:

  $\mathrm{NP}_{\mathbb{R}}$-completeness of $\mathrm{HNS}_{\mathbb{R}}$ problem; undecidability of $\mathbb{H}_{\mathbb{R}}$
- deep results concerning transfer of P versus NP results
- new framework sheds light as well on Turing results; new interesting questions arise
- difficult problems in Turing setting have easier real answers and vice versa
- etc.

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3. Recursion theory: Undecidable problems, degrees of undecidability

   some results easier: Post's problem
4. Characterization of $\mathrm{NP}_{\mathbb{R}}$ through PCPs

   new problems for algebraic computations

4. Transfer results

Since P versus NP is major question in above (and further) models as well it is natural to ask, how these (and further) questions relate in different models, in particular:

how is classical Turing complexity theory related to results over $\mathbb{R}, \mathbb{C}, \ldots$?

Transfer Results

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For all algebraically closed fields of characteristic 0 *the P versus NP question has the same answer.*

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Proof.

Main idea is to eliminate complex machine constants in algorithms for problems that can be defined without such constants;

the $\mathrm{NP}_{\mathbb{C}}$-complete problem QPS has this property;

price to pay for elimination only polynomial slowdown

Technique: Some algebraic number theory

Elimination of machine constants important technique for several transfer results; alternative proof by Koiran does it applying again Quantifier Elimination:

- algebraic constants are coded via minimal polynomials
- transcendental constants satisfy no algebraic equality test in algorithm, so each test is answered the same in a neighborhood of such a constant; using deep results from complex QE shows that there is a small rational point in such a neighborhood which can replace the transcendental constant. It can be computed without performing the QE.

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Suppose $P_{\mathbb{C}} = NP_{\mathbb{C}}$*, then* $\mathrm{NP} \subseteq \mathrm{BPP}$.

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Extract from $\mathrm{P}_{\mathbb{C}}$ algorithm for $\mathrm{QPS}_{\mathbb{C}}$ a randomized algorithm for NP-complete variant of QPS;

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Proof.

Extract from $\mathrm{P}_{\mathbb{C}}$ algorithm for $\mathrm{QPS}_{\mathbb{C}}$ a randomized algorithm for NP-complete variant of QPS; replacement of complex constants by randomly choosing small rational constants from a suitable set which with high probability contains rationals that behave the same as original constants.

Both above results not known for real algorithms; first deeper relation between real and Turing algorithms via additive real BSS machines, i.e., algorithms that only perform $+, -$ and tests $x \geq 0$;

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Theorem (Fournier & Koiran 1998)

$$\mathrm{P} = \mathrm{NP} \textit{ (Turing)} \Leftrightarrow P_{\mathbb{R}}^{add} = NP_{\mathbb{R}}^{add} \textit{ (additive model)}$$

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Proof.

Replacement of machine constants using deep result on point location in hyperplane arrangements by Meyer auf der Heide

Remark.

1. Similar results when allowing real machine constants, but introduces non-uniformity into Turing results.
2. In additive model with equality tests only, P and NP are provably different; however, questions about the polynomial hierarchy in this setting are as difficult as major open questions in classical complexity (Koiran).

3. Inside $\mathrm{NP}_{\mathbb{R}}$: Ladner's theorem in different contexts

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Theorem (Ladner 1975)

If $\mathrm{P} \neq \mathrm{NP}$ there are non-complete problems in $\mathrm{NP} \setminus \mathrm{P}$.

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Theorem (Ladner 1975)

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Proof.

Key point is diagonalization against family

$\{P_1, P_2, \ldots\}, P_i = (M_i, p_i)$ of decision machines M_i with

polynomial time bound p_i and family $\{R_1, R_2, \ldots\}, R_i = (N_i, q_i)$ of

reduction machines N_i with polynomial time bound q_i;

both families are countable and effectively enumerable in Turing

model;

Proof (cntd.)

given NP-complete $3SAT$ construct $L \in \mathrm{NP}$ s.t. one after the other for increasing i M_i fails to decide L within time bound p_i and N_i fails to reduce $3SAT$ to L within time bound q_i;

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L constructed as dimensionwise variation of 3SAT: decompose $\mathbb{N} = S \cup \bar{S}$ such that

- for inputs with length/dimension $n \in S$ L is defined as empty set and thus an easy problem;
- for inputs with length $n \in \bar{S}$ L is defined as 3SAT and thus difficult

Proof (cntd.)

Clear: if L looks like $\emptyset$ a machine from $\{R_i\}$ finally errs, if L looks like 3SAT a machine from $\{P_i\}$ finally errs

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Idea: start to fool P_1 by defining $L = 3SAT$ on dimensions $1, 2, \ldots, n_1$; here n_1 should be large enough such that there is an input formula of length at most n_1 which is decided falsely by P_1 within at most $p_1(n_1)$ steps;

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rest a folklore padding argument to force L into NP

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Turns out to be surprisingly interesting question:

- over $\mathbb{C}$ positive answer because of transfer theorem or, alternatively, model theoretic considerations; sheds also more light on why classical proof works
- over $\mathbb{R}$ surprisingly difficult and so far unsolved; partial results for restricted real models known
- leads to new research questions

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Proof.

Transfer result by Shub & Smale: 'P = NP ?' has the same answer for all algebraically closed fields of characteristic zero; efficient elimination of complex machine constants from algorithms that deal with QPS problem allows to reduce problem to the algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$, i.e., to a countable setting; then adapt Ladner's proof

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idea: consider discrete skeleton of real/complex algorithms,

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$\mathrm{P}_{\mathbb{R}}/\mathrm{const}$ is a restricted version of non-uniform class $\mathrm{P}_{\mathbb{R}}/poly$;

set of basic machines countable!

Similarly for other models: $\mathrm{P}_{\mathbb{C}}/\mathrm{const}, \mathrm{P}_{\mathbb{R}}^{\mathrm{add}}/\mathrm{const}, \mathrm{P}_{\mathbb{R}}^{\mathrm{rc}}/\mathrm{const}$

Theorem (Ben-David & M. & Michaux 2000)

If $\mathrm{NP}_{\mathbb{R}} \not\subseteq \mathrm{P}_{\mathbb{R}}/\mathrm{const}$ *there exist problems in* $\mathrm{NP}_{\mathbb{R}} \setminus \mathrm{P}_{\mathbb{R}}/\mathrm{const}$ *which are not* $\mathrm{NP}_{\mathbb{R}}$*-complete under* $\mathrm{P}_{\mathbb{R}}/\mathrm{const}$ *reductions.*

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Proof.

Construct again diagonal problem L along Ladner's line;

fool step by step all basic decision / reduction machines;

fooling dimensions computed via quantifier elimination: for each n and basic machine M running in polynomial time it is first order expressible whether M with some choice of constants decides problem upto dimension n.

Thus central: analysis of $\mathrm{P}/\mathrm{const}$ in different models;

here notions from model theory enter

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For every ω-saturated structure it is $\mathrm{P} = \mathrm{P}/\mathrm{const}$.

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Theorem (Michaux; Ben-David & Michaux & M.)

For every ω-saturated structure it is $\mathrm{P} = \mathrm{P}/\mathrm{const}$.

ω-saturation roughly means: given countable family $\phi_n(c)$ of first-order formulas such that each finite subset is commonly satisfiable, then the entire family is satisfiable.

$\mathbb{R}$ is not ω-saturated: $\phi_n(c) \equiv c \geq n$

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additive BSS over $\mathbb{R}$: $\mathrm{P}_{\mathbb{R}}^{add} = \mathrm{P}_{\mathbb{R}}^{add}/\mathrm{const}$ — thus Ladner holds

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real BSS with restricted use of constants Ladner holds

M. 2012

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input variables can be used arbitrarily; all intermediate results depend linearly on machine constants (thus no multiplication between machine constants)

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Theorem

QPS is $NP_{\mathbb{R}}^{\mathrm{rc}}$-complete (under $P_{\mathbb{R}}^{\mathrm{rc}}$-reductions)

thus: restricted model closer to full BSS model than linear/additive models $\rightsquigarrow$ motivation for studying it!

Lemma

If QPS $\not\in P_{\mathbb{R}}^{\mathrm{rc}}/\mathrm{const}$*, then there are non-complete problems in* $NP_{\mathbb{R}}^{\mathrm{rc}} \setminus P_{\mathbb{R}}^{\mathrm{rc}}/\mathrm{const}$

(i.e. above Theorem by Ben-David & M. & Michaux holds in restricted model as well)

Lemma

If QPS $\notin P_{\mathbb{R}}^{\mathrm{rc}}/\mathrm{const}$, *then there are non-complete problems in* $NP_{\mathbb{R}}^{\mathrm{rc}} \setminus P_{\mathbb{R}}^{\mathrm{rc}}/\mathrm{const}$

(i.e. above Theorem by Ben-David & M. & Michaux holds in restricted model as well)

Main proof ingredient:

Quantifier elimination possible in restricted model, thus error dimensions for $\mathrm{P}_{\mathbb{R}}^{\mathrm{rc}}/\mathrm{const}$-computations as well as -reductions effectively computable.

Theorem (M. 2012)

Ladner's theorem holds in the real BSS model with restricted use of constants.

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Ladner's theorem holds in the real BSS model with restricted use of constants.

Proof.

Main step is to prove equality $\mathrm{P}_{\mathbb{R}}^{\mathrm{rc}} = \mathrm{P}_{\mathbb{R}}^{\mathrm{rc}}/\mathrm{const}$; proof relies on a limit argument in affine geometry that allows replacement of non-uniform machine constants by uniform ones.

Let $L \in \mathrm{P}_{\mathbb{R}}^{\mathrm{rc}}/\mathrm{const}$, M corresponding basic machine using k real constants

$$E_n := \{c \in \mathbb{R}^k | M(\bullet, c) \text{ decides } \; L \cap \mathbb{R}^{\leq n}\}$$

i.e. E_n is set of suitable constants for all x upto dimension n

Proof (cntd.)

Using (among other arguments) König's lemma on infinite trees we can assume for all $n \in \mathbb{N}$:

- $E_{n+1} \subseteq E_n$, $\overline{E_{n+1}} \subseteq \overline{E_n}$
- $\bigcap_{n\in\mathbb{N}} E_n = \emptyset$
- $dim(E_n) = s$ for a fixed $s \geq 1$
- E_n is intersection of half-spaces and convex

Note: a point in $\bigcap_{n\in\mathbb{N}} E_n$ would finish the proof since it could be used as set of uniform constants for M to decide L

Proof (cntd.)

Goal: Find for each n efficiently and uniformly a choice $c^{(n)}$ of parameters such that $M(\bullet, c^{(n)})$ decides $L \cap \mathbb{R}^{\leq n}$

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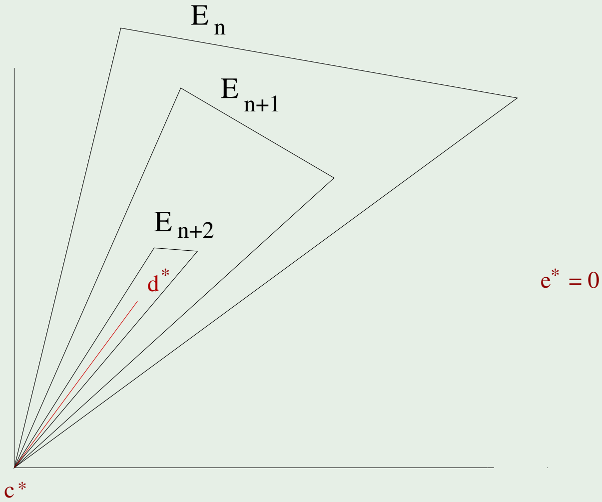
We cannot compute $c^{(n)}$, but we show existence of $c^*, d^*, e^* \in \mathbb{R}^k$ such that

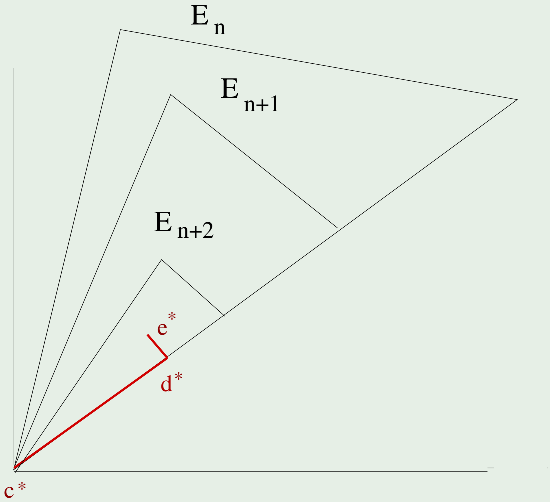
- $c^* \in \bigcap_{n \in \mathbb{N}} \overline{E_n}$
- d^* points from c^* to $\overline{E_n}$ for all n
- e^* points from $c^* + \mu_1 \cdot d^*$ to E_n for all n and $0 < \mu_1$ small enough (depending on n)

Proof (cntd.)



Limit argument: Easy case

Proof (cntd.)



Limit argument: Difficult case

Proof (cntd.)

We obtain following formula: there exist $c^*, d^*, e^* \in \mathbb{R}^k$ such that

$$\forall n \in \mathbb{N}\ \exists \epsilon_1 > 0\ \forall \mu_1 \in (0, \epsilon_1)\ \exists \epsilon_2 > 0\ \forall \mu_2 \in (0, \epsilon_2) :$$

$$c^* + \mu_1 \cdot d^* + \mu_2 \cdot e^* \in E_n\ .$$

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Finally: M's behaviour using $c^* + \mu_1 \cdot d^* + \mu_2 \cdot e^*$ for μ_1, μ_2 as above can be simulated in $\mathrm{P}_{\mathbb{R}}^{\mathrm{rc}}$

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  similar quantifier structures studied by Bürgisser & Cucker
- definition of class $\mathrm{P}_{\mathbb{R}}^{\mathrm{rc}}/\mathrm{const}$ allows certain degree of freedom; may be other definitions more helpful?

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- Ladner type result for Valiant's complexity classes obtained by Bürgisser

4. Recursion theory over $\mathbb{R}$, Post's problem

Blum-Shub-Smale: Real Halting problem is BSS undecidable

$$\mathbb{H}_{\mathbb{R}} := \{\text{code of BSS machine } M \text{ that halts on empty input}\}$$

4. Recursion theory over $\mathbb{R}$, Post's problem

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further undecidable problems:

- $\mathbb{Q}$, i.e., given $x \in \mathbb{R}$, is x rational? Problem is semi-decidable: there is an algorithms which stops exactly for inputs from $\mathbb{Q}$;
- similarly for the algebraic real numbers $\mathbb{A}$:= set of real zeros of any polynomial $p \in \mathbb{Z}[x]$;
- Mandelbrot and certain Julia sets

Typical related questions:

- degrees of undecidability
- Post's problem: are there problems easier than $\mathbb{H}_{\mathbb{R}}$ yet undecidable?
- find other natural undecidable problems equivalent to $\mathbb{H}_{\mathbb{R}}$

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Formalization of comparing problems via oracle machines:

A is Turing reducible to B iff A can be decided by a BSS machine that additionally has access to an oracle for membership in B.

A equivalent to B iff both are Turing reducible to each other

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however: no explicit problem with this property known so far

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Answer easier and more concrete over the reals!

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The rational numbers $\mathbb{Q}$ are strictly easier than $\mathbb{H}_{\mathbb{R}}$.

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Proof.

Show that $\mathbb{Q}$ is strictly easier than $\mathbb{A}$ by proving

i) $\mathbb{Q}$ is decidable with oracle for $\mathbb{A}$

ii) set $\mathbb{T} := \mathbb{R} \setminus \mathbb{A}$ of transcendent reals is not semi-decidable even with oracle for $\mathbb{Q}$

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implies claim because if $\mathbb{H}_{\mathbb{R}}$ was decidable with oracle for $\mathbb{Q}$, then $\mathbb{A}$ as well: as semi-decidable problem it can be decided with oracle $\mathbb{H}_{\mathbb{R}}$; contradiction to ii) because then also $\mathbb{T}$ is decidable with oracle $\mathbb{Q}$.

Proof (cntd.)

ad i) $\mathbb{Q}$ is decidable with oracle for $\mathbb{A}$:

function *deg* : $\mathbb{A} \mapsto \mathbb{N}_0$, $deg(a) =$ degree of algebraic number a is BSS-computable:

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function *deg* : $\mathbb{A} \mapsto \mathbb{N}_0, deg(a) =$ degree of algebraic number a is BSS-computable:

given $a \in \mathbb{A}$, try all irreducible $p \in \mathbb{Z}[x]$ for $p(a) = 0$? If p is found its degree gives $deg(a)$; has to happen because $a \in \mathbb{A}$ is assumed!

Note: irreducibility over $\mathbb{Z}[x]$ in NP (Cantor 1981), thus also decidable in BSS model

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branches: $f(x) < 0?, f(x) = 0?, f(x) > 0?, f(x) \in \mathbb{Q}?, f(x) \notin \mathbb{Q}$

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oracle computation corresponds to (possibly infinite) computation tree, in particular all $x \in \mathbb{T}$ end at a leaf;

since tree has countably many paths only there exists a finite computation path φ leading to a leaf that branches uncountably many inputs from $\mathbb{T}$: denote this set by U

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- there is no result $f_u(x) = 0$ for $x \in U$ along ϕ; otherwise test branches only finitely many points;
- there is no result $f_u(x) \in \mathbb{Q}$ along φ; no non-constant analytical f_u maps an uncountable set into $\mathbb{Q}$

Proof (cntd.)

- Theorem: For each rational f_u there is an integer D such that $f(a) \in \mathbb{Q}$ only for algebraic a of degree at most D

Continuity of the finitely many f_u and the previous theorem imply that all $x \in \mathbb{A}$ of sufficiently high degree are branched along φ as well; thus the algorithm errs on those inputs!

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There are concrete problems of the same degree as the real Halting problem:

Word problem for certain groups (M. & Ziegler 2009)

Word Problem for Groups I

Consider product bab^2ab^2aba in free semi-group $\langle\{a, b\}\rangle$;

subject to

- rule $ab = 1$ it can be simplified to b^2 but not to 1
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Fix set X and set R of equations over $\langle X\rangle = (X \cup X^{-1})^*$.

Word problem for $\langle X\rangle$: Given a formal product

$w := x_1^{\pm 1} x_2^{\pm 1 \dots} x_n^{\pm 1}, x_i \in X$, does it hold subject to R that $w = 1$?

Boone '58, Novikov '59: There exist finite X, R such that the related word problem is equivalent to discrete Halting problem.

Word Problem for Groups II

Now set $X \subset \mathbb{R}^*$ of real generators, R rules on $\langle X \rangle$;

word problem as before, but suitable for BSS setting

Example

$X := \{x_r | r \in \mathbb{R}\}; R := \{x_{nr} = x_r, x_{r+k} = x_r | r \in \mathbb{R}, n \in \mathbb{N}, k \in \mathbb{Z}\}$

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Thus this world problem is undecidable, but easier than $\mathbb{H}_{\mathbb{R}}$.

Theorem (M. & Ziegler 2009)

There are BSS decidable sets $X \subset \mathbb{R}^N$, $R \subset \mathbb{R}^*$ such that the resulting word problem is equivalent to $\mathbb{H}_{\mathbb{R}}$.

Proof.

Lot of combinatorial group theory: Nielsen reduction, HNN extensions, Britton's Lemma, amalgamation, ...

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Reals enter as index set for set of generators; no particular influence of semi-algebraic geometry; word problem is located in computational group theory and thus presents new kind of complete problem in BSS recursion theory.

Further research questions:

- power of other undecidable problems like Mandelbrot set?
- use of machine constants: what power does one gain by using more machine constants?
- find word problems representing real number complexity classes like $\mathrm{NP}_{\mathbb{R}}$ or $\mathrm{P}_{\mathbb{R}}$
- Bounded query computations: how many queries to an oracle B are needed to compute characteristic function χ_n^A for A^n on $(\mathbb{R}^*)^n$?

  Example: For $A = B = \mathbb{H}_{\mathbb{R}}$ $\log n$ queries are sufficient.

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