Lecture 13
What did we do so far ?

- We discussed definability
- Definability of classes of structures
- Definability of relations
  in one structure
- Use compactness to show
  non-definability

Outline of lecture 13

- Some history of proof systems
- Our Hilber-style system
- Many small examples
- Statement of completeness
- Outline of proof of completeness

Frege-Hilbert–style Deduction for First Order Logic

We present one deduction method for First Order Logic:

Frege-Hilbert–style deduction
(deduction sequences)

Hilbert–style deduction models to some extent **human reasoning** as practized by formally trained mathematicians.

It has its origins in the earliest textbook of First Order Logic, which was based on lectures by David Hilbert (1862-1943):

D. Hilbert and W. Ackermann, Grundzüge der theoretischen Logik, Springer 1928.

Some Pioneers of First Order Logic I

Searching for Proof Rules

Gottlob Frege (1848-1925): First Calculus of Quantifiers 1879

Giuseppe Peano (1858-1932):

David Hilbert (1862-1943):

Bertrand Russel (1872-1970):

Jacques Herbrand (1908-1931):

Gerhard Gentzen (1909-1945):

Some Pioneers of First Order Logic II

Searching for the Meaning Function

Ernst Schröder (1841-1902):

Leopold Löwenheim (1878-1957):

Thoralf Skolem (1887-1963):

C.L. Langford (1895-1964):

Kurt Gödel (1906-1978): Completeness Theorem

Alfred Tarski (1902-1983): Inventing model theory

Other Proof Systems

Natural Deduction, also called Gentzen-style Calculi, do not only formalize the use of connectives and quantification, but allow us a qualitative and quantitative analysis of proof its proof sequences.
They use **only rules** and originated in work of Gerhard Gentzen.
Frege-Hilbert–style deductions use **Axioms** and **Rules**.

Resolution is more machine friendly and is extremely popular in Automated Theorem Proving, Artificial Intelligence, Expert Systems and Logic Programming.
It originated from work by Jacques Herbrand, Martin Davis, Hilary Putnam and was popularized by J Robinson.

Frege-Hilbert-style Axioms and Inference Rules

For simplicity we define our deduction sequences only for formulas built with the connective $\rightarrow$, the logical constant $\mathbf{F}$ and the quantifier $\forall$. The remaining connectives and quantifier are used as abbreviations (macros) according to the following list:

(i) $\neg\phi$ stands for $\phi \rightarrow \mathbf{F}$;

(ii) $(\phi \vee \psi)$ stands for $((\phi \rightarrow \mathbf{F}) \rightarrow \psi)$;

(iii) $(\phi \wedge \psi)$ stands for $((\phi \rightarrow (\psi \rightarrow \mathbf{F})) \rightarrow \mathbf{F})$;

(iv) $\exists v_i \phi$ stands for $\neg\forall v_i \neg\phi$;

Definition 1 (4.6.1)

The axioms

(i) Propositional Tautologies:
Let $B(p_1, \ldots, p_n)$ be a propositional formula in $\mathbf{WFF}_{\rightarrow,\mathbf{F}}$ which is a propositional tautology. Let $\phi_1, \ldots, \phi_n$ be τ-formulas. Let ϕ be the τ-formula $B(\phi_1, \ldots, \phi_n)$ obtained from B by replacing each occurrence of the propositional variable p_i by ϕ_i. Then ϕ is an axiom.

(ii) Axiom for moving $\forall$:
If ϕ and ψ are τ-formulas, v_i be a variable which does not occur free in ϕ. Then

$$(\forall v_i(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \forall v_i \psi))$$

is an axiom.

Definition 2 (4.6.1, contd)

The axioms (contd)

(iii) Introducing Terms:

If ϕ is a τ-formula, s : **Var** $\mapsto Term(\tau)$ be a function with $s(v_i) = t$ and $s(v_j) = v_j$ for every $j \neq i$, then

$$(\forall v_i \phi(v_i) \rightarrow subst(\phi, s))$$

is an axiom, provided that no variable v_j ($j \neq i$) that occurs in t has a bound (or dummy) occurence in ϕ.

Bound and dummy variables:

In a formula $\forall v_i \phi$ we say that v_i has a **dummy occurrence** if v_i does **not occur** in ϕ.

We consider dummy occurrencies *also* as **bound occurrences**.

Why conditions on bound variables?

We look at the formula

$$\forall v_2 \exists v_1 \forall v_3 \neg R_{2,0}(v_1, v_2)$$

Here v_3 has a dummy occurrence in $\exists v_1 \neg R_{2,0}(v_1, v_2)$. Clearly

$$(\forall v_2 \exists v_1 \forall v_3 \neg R_{2,0}(v_1, v_2) \rightarrow \exists v_1 \forall v_3 \neg R_{1,0}(v_1, v_3))$$

is not valid, as it equivalent to

$$(\forall v_2 \exists v_1 \neg R_{2,0}(v_1, v_2) \rightarrow \exists v_1 \forall v_2 \neg R_{1,0}(v_1, v_2))$$

Simlarly, in the formula

$$\forall v_2 \exists v_1 \, (\neg R_{1,0}(v_1) \wedge R_{1,0}(v_2))$$

v_1 has a bound (non dummy) occurrence in $\exists v_1 \, (\neg R_{1,0}(v_1) \wedge R_{1,0}(v_2))$ and clearly

$$(\forall v_2 \exists v_1 \, (\neg R_{1,0}(v_1) \wedge R_{1,0}(v_2))$$
$$\rightarrow \exists v_1 \, (\neg R_{1,0}(v_1) \wedge R_{1,0}(v_1)))$$

is not valid.

Definition 3

The axioms (contd)

(iv) Reflexivity of Equality:
For every variable v_i

$$\forall v_i(v_i \approx v_i)$$

is an axiom.

(v) Substitutivity of Equality:
If $\phi(v_i)$ is a formula and s is the substitution $s(v_i) = v_j$ and $s(v_k) = v_k$ for $k \neq i$ and v_j has no bound occurrence in ϕ then

$$\forall v_i \forall v_j \left((v_i \approx v_j) \rightarrow (\phi \rightarrow subst(\phi, s))\right)$$

is an axiom

Soundness of the Axioms

Proposition 4

For each axiom $\psi \in FOL(\tau)$ we have that for every τ-structure $\mathfrak{A}$ and every assignement z

$$M(\psi, \mathfrak{A}, z) = 1$$

or, in other words, $\models \psi$

Proof:
By detailed computation of the meaning function, (Exercise).
For the case (i) of Propositional Tautologies the proof is the same as for Propositional Logic.
Axiom (ii) follows from proposition [4.4.5] on moving quantifiers and axiom (iii) from proposition [4.4.7] on substituting bound variables.
The axioms on equality are left as an exercise. **Q.E.D.**

Warning

In the literature there is

great variety

in the choice of the Axioms and Rules for Frege-Hilbert–style deduction systems.

They all formalize the relation

$$\Sigma \models \phi$$

(unless they are incomplete or inconsistent) and are provable (by proof sequences) one from another.

Showing this may be tricky and delicate.

Definition 5 (4.6.2)

Proof Sequences

Let Σ be a set of τ-formulas and $\phi_1, \ldots, \phi_n$ be τ-formulas. We say that $\phi_1, \ldots, \phi_n$ is a *proof sequence over* Σ if for each $i \leq n$ one of the following holds:

(i) $\phi_i \in \Sigma$;

(ii) ϕ_i is an axiom;

(iii) (Modus ponens) There are $k, l < i$ such that $\phi_l = (\phi_k \to \phi_i)$;

Proof Sequences (contd): (Generalization)

(iv) (Gen)

There is $k < i$ such that
$\phi_k = \psi(v_l)$ and
$\phi_i = \forall v_m subst(\psi(v_l), s)$,
provided that

(i) v_l not free in any of the formulas of Σ,

(ii) $s : \mathbf{Var} \mapsto \mathbf{Var}$ is a function such that
$s(v_h) = v_h$ for every $h \neq l$ and $s(v_l) = v_m$
and $m = l$
or v_m does not occur at all in ψ.

We note that $\phi_i = \forall v_l \psi(v_l)$ is allowed.
But we also allow changing the variable into something new.

Proof sequences

We write

$$\Sigma \vdash \phi$$

if there is a proof sequence

$$\phi_1, \ldots, \phi_n$$

over some $\Sigma_0 \subseteq \Sigma$ such that $\phi_n = \phi$.

Deducible formulas Soundness of Proof Sequences

Definition 6 (4.6.3)
Let Σ be a set of τ-formulas.
We define the set $Ded(\Sigma)$ by $\phi \in Ded(\Sigma)$ iff $\Sigma \vdash \phi$.

Proposition 7 (4.6.4)
If $\Sigma \vdash \phi$ then $\Sigma \models \phi$,
or, equivalently,
If $\phi \in Ded(\Sigma)$ then $\Sigma \models \phi$.

Proof:
For the case of Modus Ponens the proof is the same as for Propositional Logic.

We are left with the generalization rule
(Exercise, solution follows). **Q.E.D.**

Soundness of GEN, I

We first show that the condition
on the variables is needed. Look at

$$\exists v_1(R_{1,0}(v_2) \to \neg v_1 \approx v_2)$$

v_1 is not free, but does occur.
Clearly, it is a logical consequence that

$$\forall v_2 \exists v_1(R_{1,0}(v_2) \to \neg v_1 \approx v_2)$$

but

$$\forall v_1 \exists v_1(R_{1,0}(v_1) \to \neg v_1 \approx v_1)$$

is equivalent to

$$\exists v_1(\neg R_{1,0}(v_1) \vee \neg v_1 \approx v_1)$$

and is false in every structure $\mathfrak{A}$ with

$$(Var) \setminus \mathfrak{A}(R_{1,0}) \neq \emptyset.$$

Soundness of GEN, II

Let Σ and $\phi(v_0)$ be given and assume that for every $\mathfrak{A}$ and z be such that

$$M(\Sigma, \mathfrak{A}, z) = 1$$

also

$$M(\phi(v_0), \mathfrak{A}, z) = 1$$

This is $\Sigma \models \phi(v_0)$.

Then also for every $\mathfrak{A}$ and z with

$$M(\Sigma, \mathfrak{A}, z) = 1$$

we have

$$M(\forall v_0 \phi(v_0), \mathfrak{A}, z) = 1$$

This is $\Sigma \models \forall v_0 \phi(v_0)$, and hence
$\Sigma \cup \{\phi(v_0)\} \models \forall v_0 \phi(v_0)$. Q.E.D.

Notation 8 (4.6.5)

Let $\phi(v_i)$ be a τ–formula. We shall freely write $\phi(v_j)$ for $subst(\phi(v_i), s)$ with $s(v_i) = v_j$ and $s(v_j) = v_j$ for every $j \neq i$, when ever it should be clear from the context what is meant.

Example:

For $\phi(v_1, v_2) =$

$$\forall v_3 \, (R(v_1, v_3) \rightarrow \exists v_4 (R(v_1, v_4) \wedge R(v_2, v_4)))$$

we denote by $\phi(v_7, v_6) =$

$$\forall v_3 \, (R(v_7, v_3) \rightarrow \exists v_4 (R(v_7, v_4) \wedge R(v_6, v_4)))$$

Example 9

Particularization: $\forall v_i \phi(v_i) \vdash \phi(t)$

provided that t *is either a constant term or a new variable.*

$$\forall v_1 \exists v_2 (v_1 \approx v_2) \not\vdash \exists v_2 (F_+(v_2, c_1) \approx v_2)$$

as $\mathfrak{N} \models \forall v_1 \exists v_2 (v_1 \approx v_2)$
but $\mathfrak{N} \not\models \exists v_2 (F_+(v_2, c_1) \approx v_2)$.

Proof of Particularization:

$\forall v_i \phi(v_i)$	(1, Hypothesis)
$(\forall v_i \phi(v_i) \to \phi(t))$	(2, Axiom (iii))
$\phi(t)$	(3, MP(2))

Example 10

$\forall v_i \forall v_j \phi(v_i, v_j) \vdash \forall v_j \forall v_i \phi(v_i, v_j)$

$\forall v_i \forall v_j \phi(v_i, v_j)$	(1, Hypothesis)
$(\forall v_i \forall v_j \phi(v_i, v_j) \rightarrow \forall v_j \phi(v_i, v_j))$	(2, Axiom (iii))
$\forall v_j \phi(v_i, v_j)$	(3, MP(1,2))
$(\forall v_j \phi(v_i, v_j) \rightarrow \phi(v_i, v_j))$	(4, Axiom (iii))
$\phi(v_i, v_j)$	(5, MP(3,4))
$\forall v_i \phi(v_i, v_j)$	(6, Gen(5))
$\forall v_j \forall v_i \phi(v_i, v_j)$	(7, Gen(6))

Did we observe the conditions on variables ?

Check!

Example 11

$\forall v_i \phi(v_i) \vdash \forall v_j \phi(v_j)$

Let v_k be a variable which does not occur in ϕ at all.

$\forall v_i \phi(v_i)$	(1, Hypothesis)
$(\forall v_i \phi(v_i) \to \phi(v_k))$	(2, Axiom (iii))
$\phi(v_k)$	(3, MP(1,2))
$(\phi(v_k) \to \forall v_j \phi(v_j))$	(4, Gen(3))
$\forall v_j \phi(v_j)$	(5, MP(3,4))

Example 12

Existential Rule: $\phi(t) \to \exists v_i \phi(v_i)$

if v_i *has no bound occurrence in* ϕ *and* t *is a constant term.*

We use that $((p_1 \to \neg p_2) \to (p_2 \to \neg p_1))$
is a propositional tautology

$(\forall v_1 \neg\phi(v_1) \to \phi(t))$	(1, Axiom (iii))
$((\forall \neg\phi(v_1) \to \neg\phi(t)) \to (\phi(t) \to \neg\forall\neg\phi(v_1)))$	(2, Axiom (i))
$(\phi(t) \to \neg\forall\neg\phi(v_1))$	(3, MP(1,2))
$(\phi(t) \to \exists\phi(v_1))$	(Definition of $\exists$)

For $\phi(v_1) = (v_1 \approx v_1)$ we get

$$\vdash \exists v_2 (v_i \approx v_2)$$

Why conditions on bound variables?

We show that

$$\phi(v_i) \nvdash \forall v_i \phi(v_i)$$

Assume, for contradiction, that $\phi(v_i) \vdash \forall v_i \phi(v_i)$.

By the soundness of the Axioms and Rules we would have

$$\phi(v_i) \models \forall v_i \phi(v_i)$$

Let
$\phi(v_i) = (v_i \approx c_0)$, with $\tau = \{c_0\}$,
$\mathfrak{A}$ be a τ-structure with $\mathfrak{A}(\mathrm{Var}) = \mathbb{N}$, $\mathfrak{A}(c_0) = 0$
and z be an assignement with $z(v_i) = 0$.

Now we have $M((v_i \approx c_0), \mathfrak{A}, z) = 1$.

But there are $z' =_i z$ such that
$M((v_i \approx c_0), \mathfrak{A}, z') = 0$, a **contradiction**.

Example 13

Symmetry and Transitivity of Equality:

$t_1 \approx t_2 \vdash t_2 \approx t_1$ *and* $t_1 \approx t_2, t_2 \approx t_3 \vdash t_1 \approx t_3$

	$(t_1 \approx t_2)$	(1, Hypothesis)
$(t_1 \approx t_2 \to (t_1 \approx t_1 \to t_2 \approx t_1))$		(2, Substitutivity for $\phi(v_0, t_1) = (v_0 \approx t_1)$)
	$(t_1 \approx t_1)$	(3, Axiom)
	$(t_1 \approx t_1 \to t_2 \approx t_1)$	(4, MP(1,3))
	$t_2 \approx t_1$	(5, MP(3,4))
	– – – – – – – – – – – – – – –	
	$(t_2 \approx t_3)$	(6, Hypothesis)
$(t_2 \approx t_1 \to (t_2 \approx t_3 \to t_1 \approx t_3))$		(7, Substitutivity for $\phi(v_0, t_3) = (v_0 \approx t_3)$)
$t_1 \approx t_3$		(8-9, MP(5,7), MP(6,8))

Definition 14

Witnesses

Let $\phi(v_0, v_1, \ldots, v_n)$ be a τ-formula.

A τ-term $t(v_1, \ldots, v_n)$ is a

witness for $\exists v_0 \phi(v_0, v_1, \ldots, v_n)$
in a τ**-structure** $\mathfrak{A}$

if for every assignement z such that

$$M(\exists v_0 \phi(v_0, v_1, \ldots, v_n), \mathfrak{A}, z) = 1$$

we have also

$$M(\phi(t(v_1, \ldots, v_n), v_1, \ldots, v_n), \mathfrak{A}, z) = 1$$

Example: The term $F_+(v_0, c_1)$ is a witness for $\exists v_1 R_<(v_0, v_1)$ in $\mathfrak{N}$.

Example 15

The Witness Rule

We have used the following step informally before

(when we showed that, in an ordered field, there is no square root of -1)

There is x such that ϕ.
So let c be such an x.

c is like the personal pronoun *he* for x.

Let c be a constant symbol not in τ and let $\phi(v_i)$ be a τ-formula. Then $\phi(c) \in FOL(\tau \cup \{c\})$ and

$$\exists v_i \phi(v_1) \vdash_? \phi(c)$$

in $FOL(\tau \cup \{c\})$, provided $\exists v_i \phi(v_1)$ has no free variables.
c serves as witness for $\exists v_i \phi(v_i)$.

Exercise 16

In what sense is the Witness Rule sound (provided the condition on free variables holds) and not sound otherwise?

Can you find a proof sequence for the Witness Rule?

Example 17

Pseudo- Soundness of the Witness Rule

We have to show that

$$\exists v_i \phi(v_1) \models \phi(c)$$

So assume

$$M(\exists v_i \phi(v_i), \mathfrak{A}, z) = 1$$

Hence there is z' such that

$$M(\phi(v_i), \mathfrak{A}, z') = 1$$

We have to show

$$M(\phi(c), \mathfrak{A}, z) = 1$$

But the interpretation of c is not determined in $\mathfrak{A}$.

However, we can find such an interpretation:
We set $\mathfrak{A}(c) = z(v_i)$.

Pseudo-Soundness means just that:

Changing the vocabulary on the way.

The Witness rule is not a proof rule proper,
as we change the vocabulary on the way.

All we have is, if

$$\Sigma \vdash \exists v_i \phi(v_i)$$

then $\Sigma \cup \{\exists v_i \phi(v_i), \phi(c)\}$ is satsifiable.

We have to write the

$$\exists v_i \phi(v_i) \vdash_{witness} \phi(c_\phi)$$

instead, to mark the difference.

But it is a usefull tool to formalize informal proofs.

Strength of the Witness rule
Eliminability of the personal pronoun.

Theorem: Without proof

Assume $\phi \in FOL(\tau)$ and
$\Sigma \subseteq FOL(\tau)$ and

$$\Sigma \vdash_{witness} \phi$$

Then

$$\Sigma \vdash \phi$$

i.e. we can eliminate the use of the witness rule.

Why conditions on bound variables?

We now that

$$\forall v_1 \exists v_2 \phi(v_1, v_2) \not\models \exists v_1 \forall v_2 \phi(v_1, v_2)$$

So where is the mistake in the following ?

$\forall v_1 \exists v_2 \phi(v_1, v_2)$	(1, Hypothesis)
$\exists v_2 \phi(v_1, v_2)$	(2, Particularization, ???)
$\phi(v_1, c)$	(3, Witness, ???)
$\forall v_1 \phi(v_1, c)$	(4, Gen, ???)
$\exists v_2 \forall v_1 \phi(v_1, v_2)$	(5, Existential)

Consistency in First Order Logic

Definition 18 (4.6.8)
We say that a set of τ-formulas Σ is
inconsistent *if $\Sigma \vdash \mathbf{F}$.*
If Σ is not inconsistent, we say Σ is **consistent**.

Remark 19 (4.6.9)
By the soundness of proof sequences, if Σ is inconsistent, then Σ is not satisfiable.

Proposition 20
Σ is inconsistent iff for every ϕ $\Sigma \vdash \phi$

Manipulations of Proof Sequences I

The following are useful properties for the manipulation of proof sequences.

Proposition-Exercise 21 (4.6.10)
Let $\Sigma_0 \subseteq \Sigma$ be two sets of τ-formulas and ϕ be a τ-formula.

(i) If $\Sigma_0 \vdash \phi$ then $\Sigma \vdash \phi$;

(ii) If $\phi_1, \phi_2, \ldots, \phi_n$ is a proof sequence over Σ then for each $i \leq n$ we have that $\Sigma \vdash \phi_i$.

(iii) If $\Sigma \vdash \phi$ and $\Sigma \vdash (\phi \to \psi)$ then $\Sigma \vdash \psi$.

(iv) If $\Sigma \vdash (\phi \to (\theta \to \psi))$ and $\Sigma \vdash (\phi \to \theta)$ then $\Sigma \vdash (\phi \to \psi)$.

Manipulations of Proof Sequences II
Deduction and Dychotomy Theorem

Proposition 22 (4.6.11)
Let Σ be a set of τ-formulas and ϕ, ψ be two τ-formulas.

$$\Sigma \vdash (\phi \to \psi) \text{ iff } \Sigma \cup \{\phi\} \vdash \psi.$$

Proposition-Exercise 23 (4.6.12)
Let Σ be a set of τ-formulas and ϕ, ψ be two τ-formulas.
If both

$$\Sigma \cup \{\phi\} \vdash \psi \text{ and } \Sigma \cup \{(\phi \to \mathbf{F})\} \vdash \phi$$

then

$$\Sigma \vdash \psi$$

Manipulations of Proof Sequences III
Substitutions

Exercise 24 (4.6.14)

Let $\Sigma \subseteq FOL(\tau)$, $\phi \in FOL(\tau)$ *and*
$s : \mathbf{Var} \to Term(\tau)$ *be a substitution.*
Prove that, if $\Sigma \vdash \phi$, *then*

$$\{subst(\psi, s) : \psi \in \Sigma\} \vdash subst(\phi, s).$$

Exercise 25

Let ϕ *be a formulas with subformula* θ, *i.e.* $\phi = \alpha \circ \theta \circ \beta$, *and let* θ_1 *be formula such that* $Free(\theta) = Free(\theta_1)$ *and*

$$\theta \vdash \theta_1 \text{ and } \theta_1 \vdash \theta$$

Put $\phi_1 = \alpha \circ \theta_1 \circ \beta$. *Then*

$$\phi \vdash \phi_1 \text{ and } \phi_1 \vdash \phi$$

Hint 26

Use proposition 4.4.9.

Gödel's Completeness Theorem for Deductions **without** equality

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We denote by $FOL_{w.e.}(\tau)$ the set of τ-formulas in which the symbol $\approx$ does not occur.
We write

$$\Sigma \vdash_{w.e.} \phi$$

for the case when

- $\Sigma \subseteq FOL_{w.e.}(\tau)$, $\phi \in FOL_{w.e.}(\tau)$ and
- There is a proof sequence for ϕ using Σ in which the two axioms for equality are **not** used.

Theorem 27
Let $\Sigma \subseteq FOL_{w.e.}(\tau)$ and $\phi \in FOL_{w.e.}(\tau)$
If $\Sigma \models \phi$ then $\Sigma \vdash_{w.e.} \phi$.

Gödel's Completeness Theorem for Deductions **with** equality

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The following shows that the method of proof sequences is sufficiently powerful to obtain all logical consequences of a given set of formulas.

Theorem 28 (4.6.15)
Let Σ be a set of τ-formulas and ϕ be a τ-formula.
If $\Sigma \models \phi$ then $\Sigma \vdash \phi$.

Corollary 29 (Compactness Theorem)
Let Σ be an infinite set of τ-formulas. Σ is satisfiable iff every finite subset $\Sigma_0 \subseteq \Sigma$ is satisfiable.
This holds with and without equality.

Proof of Compactness (from Completeness)

Proof:

By the completeness theorem above Σ is satisfiable iff Σ is consistent.

Clearly, if Σ is consistent, so is every finite subset $\Sigma_0 \subseteq \Sigma$.

Conversely, assume Σ is inconsistent. By definition there is a finite subset $\Sigma_1 \subseteq \Sigma$ such that $\mathbf{F} \in Ded(\Sigma_1)$.

Therefore Σ_1 is a finite inconsistent subset. **Q.E.D.**

Proof of the Completeness Theorem

The proof of the completeness theorem comes in several stages.

- Reduction to model existence for
  consistent theories.
- Union property of consistent theories
  Like for Propositional Logic
- Maximally consistent sets
  Like for Propositional Logic
- Sets of formulas with enough witnesses.
- Model construction.
  Here we have to distinguish the cases with and without equality

Satisfiability vs. Consistency

We first observe that it suffices to prove that Σ is satisfiable iff Σ is consistent.
In other words:

Lemma 30 (4.6.18)
Assume that for every set of τ-formulas Σ, Σ is satisfiable iff Σ is consistent. Then for every set of τ-formulas Σ and every τ–formula ϕ $\Sigma \models \phi$ iff $\Sigma \vdash \phi$.

Proof:
Like for Propositiona Logic.
Use the deduction theorem. **Q.E.D.**

Union of consistent sets of sentences

Next we need a set theoretic lemma:

Lemma 31 (4.6.19)
Let $\tau = \bigcup_{i \in \mathbf{N}} \tau_i$ with $\tau_i \subseteq \tau_j$ for $i \leq j$. Let $\Sigma_i, i \in \mathbb{N}$ be a family of τ_i-formulas such that for each $i \in \mathbb{N}$ Σ_i is consistent and $\Sigma_i \subseteq \Sigma_{i+1}$. Then $\Sigma = \bigcup \Sigma_i$ is consistent.

Proof:
Like for Propositional Logic.
Σ is consistent. For, otherwise, there is a finite deduction sequence showing inconsistency over some finite subset $X \subseteq \Sigma$. But then there is an $i \in \mathbb{N}$ with Σ_i inconsistent. **Q.E.D.**

Maximally consistent sets

Definition 32 (4.6.20)
Let Σ be a set of τ–sentences.
Σ *is* maximally consistent *if Σ is consistent and for every τ–sentence ϕ*

$$\phi \in \Sigma \text{ or } \neg\phi \in \Sigma.$$

Lemma 33 (4.6.21)
Let Σ be a consistent set of τ–sentences. Then there is a maximally consistent set of τ–sentences Σ^* with $\Sigma \subseteq \Sigma^*$.

Construction of maximally consistent sets

Proof:
Let $\{\phi_i\}$ be an enumeration of all τ–sentences.

Let $\Sigma_0 = \Sigma$.
Let $\Sigma_{n+1} = \Sigma_n \cup \{\phi_n\}$ if it is consistent,
and $\Sigma_{n+1} = \Sigma_n \cup \{\neg\phi_n\}$ otherwise.

Use the Dychotomy theorem to show that for each $n \in \mathbb{N}$ Σ_n is consistent.

Now let $\Sigma^* = \bigcup_{i\in\mathbb{N}} \Sigma_n$ be the
union of all the Σ_n.
Σ^* is consistent by lemma 31.

To show that Σ^* is maximally consistent, let ψ be a τ–sentence. $\psi = \phi_k$ for some $k \in \mathbb{N}$. But then either ψ or $\neg\psi$ is in Σ_{k+1} and therefore in Σ^*. **Q.E.D.**

Examples of maximally consistent sets, I (4.6.22)

Example 34
Let $\mathfrak{A}$ be a τ-structure and

$$Th(\mathfrak{A}) = \{\phi \in SENT(\tau) : \mathfrak{A} \models \phi\}$$

$Th(\mathfrak{A})$ is maximally consistent.

Proof:
To see this we note first that $Th(\mathfrak{A})$ is satisfiable (by $\mathfrak{A}$), and therefore consistent, by the soundness of the deduction rules.

Now let ϕ be a τ-sentence.
Clearly, either $\mathfrak{A} \models \phi$ or $\mathfrak{A} \models \neg\phi$.
Therefore either $\phi \in Th(\mathfrak{A})$ or $\neg\phi \in Th(\mathfrak{A})$, which shows that $Th(\mathfrak{A})$ is maximally consistent. **Q.E.D.**

Examples of maximally consistent sets, II (4.6.22, contd)

Definition 35
We write $\mathfrak{A} \equiv \mathfrak{B}$, *and say that* $\mathfrak{A}$ *is* elementarily equivalent to $\mathfrak{B}$ *if for every* τ*-sentence* ϕ *we have that* $\mathfrak{A} \models \phi$ *iff* $\mathfrak{B} \models \phi$.

Example 36
Let Σ *be a consistent set of* τ*–sentences such that*
for every two τ*-structures* $\mathfrak{A}$, $\mathfrak{B}$ *with* $\mathfrak{A} \models \Sigma$ *and* $\mathfrak{B} \models \Sigma$ *we have that* $\mathfrak{A} \equiv \mathfrak{B}$.
Then $Ded(\Sigma)$ *is maximally consistent.*

(Exercise, similar to the above).

Having enough witnesses, I

Definition 37 (4.6.23)
A constant τ–term *is a τ–term without free variables.*

Definition 38 (4.6.24)
Let Σ be a set of τ–sentences.
We say that Σ has enough witnesses
if for every τ–formula $\phi(v_i)$ with v_i the only free variable in ϕ there is a constant τ–term t such that if $\exists v_i \phi(v_i) \in \Sigma$ then $\phi(t) \in \Sigma$.

Recall that $\phi(t)$ is an abbreviation for the result of substituting t for v_i in ϕ.

Having enough witnesses, II

Lemma 39 (4.6.25)
Let Σ be a countable consistent set of τ–sentences. Then there is a countable vocabulary τ^+ and a countable consistent set of τ^+-sentences Σ^+ which has enough witnesses.

Proof:
Enumerate all the formulas with one free variable.

Now, use the Choice rule for each of the formulas by introducing a new constant. **Q.E.D.**

Having enough witnesses, III

Lemma 40 (4.6.26)
Let Σ be a countable consistent set of τ–sentences. Then there is a countable vocabulary $\tau^{\#}$ and a a countable set of $\tau^{\#}$-sentences $\Sigma^{\#}$ which has enough witnesses and is maximally consistent.

Proof:
Use lemma 33 and lemma 39. **Q.E.D.**

Constructing a model (without equality)

Lemma 41 (4.6.27)
Let Σ be a countable set of τ–sentences not containing the equality symbol. If Σ has enough witnesses and is maximally consistent, then Σ is satisfiable.

Proof:
Use the set of witnesses and constant terms as the universe of a model.
Interpret the function symbols by themselves.
Interpret the relation symbols according to whether $R_{n,\alpha}(t_1, \ldots, t_n) \in \Sigma$. **Q.E.D.**

Proof of the completeness theorem (without equality)

The proof of the completeness theorem now proceeds as follows.

- Assume Σ is consistent.
- Using, lemma 40 we can find Σ^* with $\Sigma \subseteq \Sigma^*$, such that Σ^* is maximally consistent and has enough witnesses.
- Using lemma 41 we conclude there is a τ^*–structure $\mathfrak{A}$ with $\mathfrak{A} \models \Sigma^*$.
- Regarding $\mathfrak{A}$ as a τ–structure we conclude that $\mathfrak{A} \models \Sigma$.

Proof of the Completeness Theorem (with equality), I

- We build the model starting with the set of constant terms T.
- We define an equivalence relation between constant terms by

$$t_1 \sim t_2 \text{ iff } t_1 \approx t_2 \in \Sigma^*$$

- The universe of our model $\mathfrak{A}$ is the set

$$\{[t] : t \in T\}$$

- We define the function $\mathfrak{A}(F_{2,0})$ by setting

$$\mathfrak{A}(F_{2,0})([t_1], [t_2]) = [F_{2,0}(t_1, t_2)]$$

and have to show that this is well defined.

Proof of the Completeness Theorem (with equality), II

- We have to show that for
  $t_1 \sim t_3$ and $t_2 \sim t_4$ we have

  $$\mathfrak{A}(F_{2,0})([t_1], [t_2]) = \mathfrak{A}(F_{2,0})([t_3], [t_4])$$

  Here the axioms of equality are used.

- We define the relation $\mathfrak{A}(R_{2,0})$ by setting

  $$([t_1], [t_2]) \in \mathfrak{A}(R_{2,0}) \text{ iff } R_{2,0}(t_1, t_2) \in \Sigma^*$$

- Finally we show that $\mathfrak{A} \models \Sigma^*$
  This is similar to the case without equality.