Lecture 1

A landscape of graph parameters

and graph polynomials

Outline of Prague Lecture 1

- Introducing graph polynomials
- The chromatic polynomial
- The characteristic polynomial
- The matching polynomials
- Multivariate graph polynomials: The Tutte polynomial
- Complete graph invariants
- Comparing graph invariants: Getting started
- Comparing graph invariants: Towards a general theory
- Semantic vs syntactic properties of graph parameters
- Thanks

Graph isomorphims *

Let $\mathcal{DG}$ be the class of finite graphs $\langle V(G), E(G)\rangle$ where $V + V(G)$ is a finite set and $E + E(G) \subseteq V^2$ is a set of (directed edges). $G \in \mathcal{DG}$ is called a directed graph. $\mathcal{G}$ be the class of finite graphs, i.e. where E is symmetric.

For $G_1, G_2 \in \mathcal{DG}$ $f : G_1 \to G_2$ is an **isomorphisms** if

(i) f is a bijection, and

(ii) For $u, v \in V(G_1)$ we have
$(u, v) \in E(G_1)$ iff $(f(u), f(v)) \in E(G_2)$.

G_1 **and** g_2 **are isomorphic**, denoted by $G_1 \simeq G_2$,
if there is an isomorphism $f : G_1 \to G_2$.

Rings $\mathcal{R}$

Let $\mathcal{R}$ a ring.

- $\mathcal{R} = \mathcal{B}_2$ the two element bolean ring.
- $\mathcal{R} = \mathbb{Z}_2$ the two element field.
- $\mathcal{R} = \mathbb{Z}$, the ring of integers.
- $\mathcal{R} = \mathbb{Z}[X]$, the polynomial ring over the integers with one indeterminate.
- $\mathcal{R} = \mathbb{Z}[X_1, \ldots, X_k]$, the polynomial ring over the integers with k indeterminates.
- $\mathcal{R} = \mathbb{R}$, the ring of real numbers.

Definition 1 *Graph invariants over a ring $\mathcal{R}$*

Let $\mathcal{R}$ a ring, $\mathcal{G}$ the class of finite graphs.
A function

$$f : \mathcal{G} \to \mathcal{R}$$

is a graph invariant if for any two isomorphic graphs G_1, G_2 we have $f(G_1) = f(G_2)$.

Example 2 *Boolean graph invariants*

Here the ring is $\mathcal{B}_2$,
or any ring $\mathcal{R}$, but the values of the invariant are either 0 or 1.

- Connectedness
- Regular, or regular of degree r.
- Any First Order expressible graph property.
- Any Second Order expressible graph property.
- Belonging to any specific class of graph closed under isomorphisms.
- There are continuum many boolean graph invariants.

Example 3 *Numeric graph invariants*

Here the ring is $\mathbb{Z}$.

- The cardinality of $V(G)$ or $E(G)$.
- The number of connected components of G, usually denoted by $k(G)$.
- The coloring number of G.
- The size of the maximal clique (independent set).
- The diameter of G.
- The radius of G.
- The minimum length of a cycle in G, if it exists, called the girth of the graph G.

Example 4 *Graph polynomials*

Here the ring is $\mathbb{Z}[X]$.

The graph polynomial $p(G, X)$ gives for each value of X a graph invariant, hence it encodes a possibly infinite family of graph invariants.

The study of graph polynomials has a long history concentrating on particular polynomials.

The **classic** and very readable book is:

- Norman Biggs
  Algebraic Graph Theory
  Cambridge University Press
  1974 (2nd edition 1993)

Example 5 *The chromatic polynomial* *

- Let $\chi(G, X)$ denote the number of vertex colorings of G with X colors.
  We shall prove that $\chi(G, X)$ is a polynomial in X,
  called the **chromatic polynomial of** G.

The chromatic polynomial was first introduced by G.D. Birkhoff in 1912.

It led to a very rich theory, although it was introduced
in a (failed) attempt to prove the 4-color conjecture.

The most comprehensive monograph about the chromatic polynomial is

- F.M. Dong, K.M. Koh and K.L. Teo
  Chromatic polynomials and chromaticity of graphs
  World Scientific, 2005

What can we do with a graph polynomial?

- Study its zeros.
- Interpret its coefficients in various normal forms.
- Interpret its evaluations.
- Study graphs uniquely determined by the polynomial.
- Study graph classes having the same graph polynomial.
- Study the strength of the graph invariant.

Digression 1:

Typical theorems about the chromatic polynomial

Skip digression

Theorem 1 (G. Birkhoff, 1912)

$\chi(G, X)$ is indeed a polynomial in X of degree $|V(G)|$.

Proof Let $e = (a, b)$ be an edge of the graph G. $G - e$ and G/e are obtained from G by deleting, respectively contracting the edge e.
We use induction over $E(G)$.

- First we observe that for disjoint unions $G = G_1 \sqcup G_2$ we have $\chi(G, X) = \chi(G_1, X) \cdot \chi(G_2, X)$.
- For n isolated points $\bar{K_n}$ we have $\chi(\bar{K_n}, X) = X^n$.
- $\chi_{a \neq b}(G, X)$ is the number of X-colorings of G with a and b having different colors.
- $\chi_{a=b}(G, X)$ is the number of X-colorings of G with a and b having the same color.
- $\chi(G - e, X) = \chi_{a \neq b}(G - e, X) + \chi_{a=b}(G - e, X) = \chi(G, X) + \chi(G/e, X)$
- $\chi(G, X) = \chi(G - e, X) - \chi(G/e, X)$ Q.E.D.

Normal forms of $\chi(G, X)$, I

As $\chi(G, X)$ is a polynomial we can write it as

$$\chi(G, X) = \sum_{i}^{|V(G)|} b_i(G) X^i$$

For the disjoint union we noted that

Proposition 2

$$\chi(G_1 \sqcup G_2, X) = \chi(G_1, X) \cdot \chi(G_2, X).$$

Normal forms of $\chi(G, X)$, II

We define $X_{(i)} = X \cdot (X-1) \cdot \ldots \cdot (X-i+1)$.
We write

$$\chi(G, X) = \sum_{i}^{|V(G)|} c_i(G) X_{(i)}$$

We define a an operation $\circ$ on the $X_{(i)}$ by $X_{(i)} \circ X_{(j)} = X_{(i+j)}$ and extend it naturally to polynomials in $X_{(i)}$.

The join of two graphs G_1, G_2, $G_1 + G_2$,
is obtained by taking the disjoint union and
adding all the edges between $V(G_1)$ and $V(G_2)$.

Theorem 3

$$\chi(G_1 + G_2, X) = \left(\sum_{i}^{|V(G)|} c_i(G_1) X_{(i)} \circ \sum_{i}^{|V(G)|} c_i(G_2) X_{(i)} \right)$$

Trees and tree-width

- For trees T with n vertices we have $\chi(T,X) = X \cdot (X-1)^{n-1}$.
  I particular, any two trees on n vertices have
  the same chromatic polynomial.

- (R. Read, 1968)
  Conversely, for G a simple graph, if $\chi(G,X) = X \cdot (X-1)^{n-1}$, then G is a tree.

- (C. Thomassen, 1997)
  If G has tree-width $k \geq 2$ then for every real number $a > k$ we have $\chi(G,a) \neq 0$.

- (B. Courcelle, J.A. Makowsky, U. Rotics, 2000)
  For graphs G with tree-width at most k the polynomial $\chi(G,X)$ can be computed in polynomial time.

- (J.A. Makowsky, U. Rotics, 2005)
  For graphs G with clique-width at most k the polynomial $\chi(G,X)$ can be computed in polynomial time.

Planar graphs and the chromatic polynomial.

Theorem 4 (P.J. Heawood, 1890)
Every planar graph is 5-colorable.
$\chi(G, 5) \neq 0$ *for* G *planar.*

Theorem 5 (G. Birkhoff and D. Lewis, 1946)
$\chi(G, a) \neq 0$ *for* G *planar and* $a \in \mathbb{R}, a \geq 5$.

Note that this is much stronger than the 5-color theorem.

Theorem 6 (K. Appel and W. Haken, 1977)
Every planar graph is 4-colorable.
$\chi(G, 4) \neq 0$ *for* G *planar.*

Problem 7
Find an analytic proof of the 4-color theorem.

Conjecture 8 (G. Birkhoff and D. Lewis, 1946)
For G *planar, there are no real roots of* $\chi(G, a)$ *for* $4 \leq a \leq 5$.

Real roots of $\chi(G, X)$

We note that $\chi(G, 0) = 0$ always, and $\chi(G, 1) = 0$ any graph with at least one edge.

Theorem 9 (D. Woodall, 1977)
Let G be any graph.

- *There are no negative real roots of $\chi(G, X)$.*
- *There are no real roots of $\chi(G, X)$ in the open interval $(0, 1)$.*

Theorem 10 (B. Jackson, 1993)

- *There are no real roots of $\chi(G, X)$ in the semi-open interval $(1, \frac{32}{27}]$.*
- *For any $\epsilon > 0$ there is a graph G_ϵ such that $\chi(G_\epsilon, X)$ has a root in $(\frac{32}{27}, \frac{32}{27} + \epsilon)$.*

Theorem 11 (S. Thomassen, 1997)
For any real numbers a_1, a_2 with $\frac{32}{27} \leq a_1 < a_2$ there exists a graph G such that $\chi(G, X) = 0$ for some $a \in (a_1, a_2)$.

Other counting interpretations: Acyclic orientations

An **orientation** of a graph G is a function which for every edge $e = (a, b)$ selects a source value $s(e) \in \{a, b\}$

An orientation is **acyclic**, of there are no oriented cycles.

Theorem 12 (R.P. Stanley, 1993)
The number of acyclic orientations of a graph G is given by the absolute value $|\, \chi(G, -1)\,|$.

Subgraph expansions

Let G be a graph with $k(G)$ connected components.
Let $S \subset E(G)$ and denote by $\langle S \rangle$ the subgraph generated by S in G.

- The **rank** $r(G)$ is defined as $r(G) = \mid V(G) \mid - k(G)$.
- The **corank** $s(G)$ is defined as $s(G) = \mid E(G) \mid - \mid V(G) \mid + k(G)$.
- The **rank polynomial** of a graph is defined by

$$R(G; X, Y) = \sum_{S \subseteq E(G)} X^{r(\langle S \rangle)} Y^{s(\langle S \rangle)}$$

Theorem 13 (H. Whitney, 1932)

(i) $\chi(G, X) = \sum_{S \subseteq E(G)} (-1)^{|S|} X^{|V(G)| - r(\langle S \rangle)}$

(ii) $\chi(G, X) = X^{|V|} R(G, -X^{-1}, -1)$

End of digression on typical theorems about the chromatic polynomial

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Example 6 *The characteristic polynomial* *

- Let $V = [n]$ and let A_G be the (symmetric) adjacency matrix of G with $(A)_{j,i} = (A)_{i,j} = 1$ iff there is an edge between vertex i and vertex j.

- We denote by $P(G, X)$ the polynomial

  $$\det(X \cdot \mathbf{1} - A)$$

  $P(G, X)$ is a graph invariant and a polynomial in X, called the **characteristic polynomial of** G.

- The set of roots of $P(G, X)$ (with multiplicities) are the eigenvalues of A_G, and are called the **spectrum of the graph** G.

The characteristic polynomial and the spectrum of a graph was first studied in the 1950ties

T.H. Wei 1952, L.M. Lihtenbaum 1956,
L. Collatz and U. Sinogowitz 1957,
H. Sachs 1964, H.J. Hoffman 1969

The characteristic polynomial: Literature

The characteristic polynomial and spectra of graphs have a very rich literature with important applications in chemistry under the name **Hückel theory**.

- N. Biggs, Algebraic Graph Theory,
  Cambridge University Press, 1994 (2nd edition)

- D.M. Cvetković, M. Doob and H. Sachs
  Spectra of Graphs
  Johann Ambrosius Barth, 1995 (3rd edition)

- D.M. Cvetković, P. Rowlinson and S. Simić
  Eigenspaces of Graphs
  Encyclopedia of Mathematics, vol. 66
  Cambridge University Press, 1997

- N. Trinajstić
  Chemical Graph Theory
  CRC Press, 1992 (2nd edition)

Digression 2:

Typical theorems about the characteristic polynomial

Skip digression

Coefficients of $P(G, X)$

We write

$$P(G, X) = \sum_{i=0}^{|V(G)|} c_i(G) \cdot X^{n-i}$$

Proposition 14

(i) $c_0 = 1$

(ii) $c_1 = 0$

(iii) $-c_2 = \mid E(G) \mid$ *is the number of edges of* G*.*

(iv) $-c_3$ *is twice the number of triangles of* G*.*

Eigenvalues of G, I

As in linear algebra, the zeros of $P(G, X)$ are called **eigenvalues of the matrix** A_G, or **eigenvalues of the graph** G,

Proposition 15

(i) *All the eigenvalues of G are real.*

(ii) *If G is connected, the largest eigenvalue of G has multiplicity 1.*

(iii) *If G is connected and of diameter at least d, the G has at least $d + 1$ distinct zeros.*

(iv) *The complete graph is the only connected graph with exactly two distinct eigenvalues, $P(K_n, X) = (X + 1)^{n-1}(X - n + 1)$.*

(v) *Let $\Lambda(G)$ be the largest eigenvalue of G. G is bipartite iff $-\Lambda(G)$ is also an eigenvalue of G.*

Eigenvalues of G, II

Proposition 16
Let G be a regular graph of degree r. Then

(i) r *is an eigenvalue of* G

(ii) *If G is connected, then the multiplicity of r is* 1.

(iii) *For any eigenvalue λ of G we have $\mid \lambda \mid \leq r$.*

(iv) *The multiplicity of the eigenvalue r is the number of connected components of G.*

Eigenvalues of G, III

$\lambda(G)$ denotes the smallest eigenvalue of G.
$\lambda_2(G)$ denotes the second largest eigenvalue of G.
$\Lambda(G)$ denotes the largest eigenvalue of G.

Proposition 17

(i) *If H is an induced subgraph of G, then $\lambda(H) \le \lambda(G)$.*

(ii) *If H is an induced subgraph of G, then $\Lambda(H) \le \Lambda(G)$. If H is a proper induced subgraph, then $\Lambda(H) < \Lambda(G)$.*

(iii) *For no graph G is $\lambda(G) \in (-1, 0)$.*

(iv) *Let G have at least two vertices. $\lambda(G) = -1$ iff G is a complete graph.*

(v) *For no graph G is $\lambda(G) \in (-\sqrt{2}, -1)$.*

(vi) *(J. Smith, 1970) $\lambda_2(G) \le 0$ iff G is a complete multipartite graph.*

End of digression on typical theorems about the characteristic polynomial

Example 7 *The acyclic or matching defect polynomial, I* *

We denote by $m_k(G)$ the number of k-matchings of a graph G, with $m_0(G) = 1$ by convention.

- The polynomial

$$m(G, X) = \sum_{k}^{\frac{n}{2}} (-1)^k m_k(G) X^{n-2k}$$

is called the **acyclic polynomial of** G and also the **reference polynomial** or **matching defect polynomial**.

The acyclic or matching defect polynomial, II

The acyclic polynomial has important applications in Chemistry (Hückel theory again) and and Molecular Physics of Ferromagnetisms. It was first studied in the 1970 (Heilman and Lieb, Kunz)

- L. Lovász and M.D. Plummer
  Matching Theory
  Annals of Discrete mathematics, vol. 29
  North-Holland 1986

- N. Trinajstić,
  Chemical Graph Theory
  CRC, 1992 (2nd edition)

- P.J. Garratt
  Aromaticity
  John Wiley and Sons, 19xx

Example 8 *The matching (generating) polynomial*

- The polynomial

$$g(G, X) = \sum_{k}^{n} m_k(G) X^k$$

is called the **matching polynomial of** G or the **matching generating polynomial of** G.

- It is easy to verify the identity

$$m(G, X) = X^n g(G, (-X^{-2}))$$

Example 9 *Multi-variate graph polynomials* *

Inspired by H. Whitney's work (1932) W.T. Tutte (1947, 1954) investigated generalizations of the chromatic polynomial to a polynomial in two variables, which he called the **dichromatic polynomial**, but now is called the **Tutte polynomial**, $T(G, X, Y)$.

The Tutte polynomial and its many generalizations became prominent, due to its many combinatorial interpretations in fields outside graph theory:

- Knot theory (via the Jones polynomial and its relatives)
- Statistical mechanics
- Quantum theory and quantum computing
- Chemistry

Example 10 *The Tutte polynomial*

Let $G = (V, E)$ be a graph,
and for $A \subseteq E$, let $G_A = (V, A)$ be a spanning subgraph.

The rank $r(G; A)$ is defined as $|\, V(G) \,| - k(G_A)$.

The **Tutte polynomial of** G is defined as

$$T(G; X, Y) = \sum_{A \subseteq E} (X - 1)^{r(G;E) - r(G;A)} \cdot (Y - 1)^{|A| - r(G;A)}$$

This looks confusing and innocent at the same time.

The fascination with the Tutte polynomial

The Tutte polynomial is like
a **magician's hat** with
rabbits, birds and other surprises coming out.

Easy manipulations produce various combinatorial counting functions. We have, at first glance surprisingly, the following

- $T(G, 1, 1)$ counts the number of spanning trees of G.
- $T(G, 2, 1)$ counts the number of forests of G.
- $T(G, 2, 0)$ counts the number of acyclic orientations of G.
- The chromatic polynomial is given by
$$\chi(G, X) = (-1)^{r(G;E)} X^{k(G)} T(G; 1 - X, 0)$$
- The reliability polynomial and the flow polynomial can also be derived with similar formulas.

Complete graph invariants *

Skip complete graph invariants

Complete graph polynomials

A graph invariant f is graph-complete if for any two graphs G_1, G_2 with $f(G_1) = f(G_2)$ we have also $G_1 \simeq G_2$.

Are there complete graph polynomials?

The following is a graph-complete graph invariant.

- Let $X_{i,j}$ and Y be indeterminates.
  For a graph $\langle V, E \rangle$ with $V = [n]$ we put

$$Compl(G, Y, \bar{X}) = Y^{|V|} \cdot \left(\sum_{\sigma \in \mathfrak{S}_n} \prod_{(i,j) \in E} X_{\sigma(i), \sigma(j)} \right)$$

  Here $\mathfrak{S}_n$ is the permutation group of $[n]$.

Challenge: Find a polynomial in a constant finite number of indeterminates which is a graph-complete graph invariant.

An “unnatural” graph-complete invariant

Let $g : \mathcal{G} \rightarrow \mathbb{N}$ be a Gödel numbering for labeled graphs of the form $G = \langle [n], E, <_{nat} \rangle$.

We define a graph polynomial using g:

$$\Gamma(G, X) = \sum_{H \simeq G} X^{g(H)}$$

Clearly this is a graph invariant.

But it is **“obviously unnatural”** !

Can we make precise

what a **natural** graph polynomial should be?

Complete graph invariants skipped

Comparing graph invariants: Getting started *

In the literature we often find statements or questions of the form

- The Tutte polynomial is generalization of the chromatic polynomial.
- The Tutte polynomial does not determine the matching polynomial.
- Is there a natural most general graph polynomial?

We attempt to make this precise

Definition 11

Induced graph invariants

Let $\mathcal{H} \subseteq \mathcal{G}$ be a class of graphs closed under isomorphisms.
Let F be a set of graph invariants in a ring $\mathcal{R}$,
and let g be one more graph invariant.

We say that F **induces** g **on** $\mathcal{H}$,
or g **is a consequence of** F,
if for any two graphs $G_1, G_2 \in \mathcal{H}$ such that $f(G_1) = f(G_2)$ for all $f \in F$
we also have $g(G_1) = g(G_2)$.

We denote by $Ind_{\mathcal{R}}^{\mathcal{H}}(F)$ the set of graph invariants in $\mathcal{R}$ induced by F on $\mathcal{H}$.

We write also $F \models_{\mathcal{R}}^{\mathcal{H}} g$ for $g \in Ind_{\mathcal{R}}^{\mathcal{H}}(F)$.

How do we see if $F \models_{\mathcal{R}}^{\mathcal{H}} g$?

Example 12

Algebraically derived invariants

Let f, g be two graph invariants in $\mathcal{R}$.
Then the following are derived invariants of $F = \{f, g\}$:

- $f + g, f - g, f \times g$
- The formal derivative f'.
- Let $\phi : \mathcal{R}^2 \rightarrow \mathcal{R}$ be a function.
  Then $\phi(f, g)$ is induced by F.

More examples of induced graph invariants

- Invariants induced by the characteristic polynomial
- Invariants induced by the acyclic (matching) polynomial
- Invariants induced by the chromatic polynomial
- The acyclic polynomial and the characteristic polynomial
- The acyclic polynomial and the chromatic polynomial
- The chromatic polynomial and Tutte polynomial
- The Tutte polynomial and the matching polynomials

Skip Examples

Examples 13

Invariants induced by the characteristic polynomial

The characteristic polynomial $P(G, X)$ induces

- The number of vertices $\mid V \mid$.
- The number of edges $\mid E \mid$.
- The number of triangles of G.

We also have $P(K_{1,4}, X) = P(C_4 \sqcup E_1, X)$
but $K_{1,4}$ has no 2-matchings, whereas C_4 does.
Hence the $P(G, X)$ does not induce the number of connected components $k(G)$ nor $m(G, X)$.

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Example 14

Invariants induced by the acyclic (matching) polynomial.

The acyclic polynomial $m(G, X)$ induces

- The number of vertices $\mid V \mid$.
- The number of edges $\mid E \mid$.
- The number of perfect matchings.
- the matching generating polynomial.

On the otherside $m(E_n, X) = 1$ for all $n \in \mathbb{N}$,
whereas $P(E_n, X) = X^n$.
Hence the $m(G, X)$ does not induce the characteristic polynomial $P(G, X)$.
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Example 15

Invariants induced by the chromatic polynomial

The following are induced by $\chi(G, X) = \sum_{i=1}^{n} (-1)^{n-i} h_i X^i$:

- The cardinality of $V(G) = n$ is the degree of $\chi(G, X)$.
- The cardinality of $E(G) = m = h_{n-1}$.
- The chromatic number $\chi(G)$ is the smallest integer a such that $\chi(G, a) > 0$.
- The number of connected components $k(G)$ is the multiplicity of zeros $X = 0$.
- The number of blocks $b(G)$ is the multiplicity of zeros $X = 1$.
- The girth $g = g(G)$ is given by the fact that for $0 \leq i \leq g - 2$ we have $h_{n-i} = \binom{E(G)}{i}$.

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Example 16

The acyclic polynomial and the characteristic polynomial.

Theorem 17 (I. Gutman, 1977)

$P(G, X) = m(G, X)$ *iff* G *is a forest.*

For $\mathcal{H} = \mathcal{F}$ *the forests we have*

$$P(G, X) = m(G, X)$$

i.e., the acyclic (matching defect) polynomial and the characteristic polynomial coincide,
and we have

$$P(G, X) \models^{\mathcal{F}} m(G, X) \text{ and } m(G, X) \models^{\mathcal{F}} P(G, X).$$

and

$$P(G, X) \models^{\mathcal{F}} g(G, X) \text{ and } g(G, X) \models^{\mathcal{F}} P(G, X).$$

In general, none induces the other.

I. Gutman, The acyclic polynomial of a graph, Publ. Inst. Math. (Beograd) (N.S.) 22 (36) (1977), pp. 63-69.

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Example 18

Adjoint polynomials

Definition 19
The complement graph of the simple graph $G = (V, E)$
is the graph $\bar{G} = (V, V^2 - D(V) - E)$ *.*

For a graph polynomial $g = g(G, \bar{X})$ *the adjoint polynomial* $\hat{g}(G, \bar{X})$ *of* g *is defined by* $\hat{g}(G, \bar{X}) = g(\bar{G}, \bar{X})$*.*

WARNING: In the literature on the chromatic polynomial the definition of adjoint polynomials differs!

The acyclic polynomial and the chromatic polynomial.

Theorem 20 (E.J. Farrell and E.G. Whitehead Jr. 1992)
For $\mathcal{H} = \mathcal{TF}$, the triangle free graphs, we have

$$\hat{\chi}(G, X) \models^{\mathcal{TF}} m(G, X) \text{ and } m(G, X) \models^{\mathcal{TF}} \hat{\chi}(G, X).$$

*i.e., the acyclic (matching defect) polynomial
and the adjoint chromatic polynomial mutually induce each other.*

*Note that $\chi(P_4) = \chi(K_{1,3})$, $P_4 \simeq \bar{P}_4$, but $m(P_4) \neq m(K_{1,3})$. On the other hand, $m(E_n) = 1$ for each $n \in \mathbb{N}$, and $\chi(E_n) = X^n$.
Hence, in general, none induces the other.*

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Example 21

The chromatic polynomial and Tutte polynomial

(i) The chromatic polynomial $\chi(G, X)$ is not induced by the Tutte polynomial $T(G, X, Y)$.

(ii) On connected graphs $\mathcal{C}$ we have $T(G, X, Y) \models^{\mathcal{C}} \chi(G, X)$ for

(iii) Tutte polynomial $T(G, X, Y)$ is not induced by the the chromatic polynomial $\chi(G, X)$.

Proof:

(i) Let E_n be the graph with n vertices and no edges. We have $T(E_n, X, Y) = 1$ but $\chi(E_n, X) = X^n$.

(ii) (After W.T. Tutte, 1954) $\chi(G, X) = (-1)^{|V|-k(G)} X^{k(G)} T(G, 1 - X, 0)$.

(iii) (After M. Noy, 2003) Let W_n be the wheel with n spokes. It is known that $T(G, X, Y) = T(W_n, X, Y)$ implies that $G \simeq W_n$ for all n.

But there is a $G \not\simeq W_5$ with $\chi(G, X, Y) = \chi(W_5, X, Y)$.

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Example 22

The Tutte polynomial and the matching polynomials

- The matching polynomial is not induced by the Tutte polynomial, even on connected planar graphs.

- The Tutte polynomial is not induced by the matching polynomial, even on connected planar graphs.

Proof:

(i) For trees with n vertices t_n we have $T(t_n, X, Y) = X^{n-1}$. But it is easy to see that $K_{1,n-1}$ and P_n are both trees with n vertices and their matching polynomials differ, as $K_{1,n-1}$ has no 2-matching but P_n has for $n \geq 3$.

(ii) On the other hand $C_3 \sqcup_e C_5$ and $C_4 \sqcup_e C_4$ have the same matching polynomials (check by hand) but have different Tutte polynomials, as the Tutte polynomials counts cliques of given size.

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What do we learn?
What do we ask?

- Polynomial graph invariants are still a mystery.

- Can we analyze the consequence relation for polynomial invariants?

- Can we identify "good invariants"?

- What are appropriate complexity classes for graph invariants?

Comparing graph parameters and graph polynomials

Towards a general theory

Jointly prepared with E.V. Ravve

Back to outline

Graph parameters and graph polynomials

Let $\mathcal{R}$ be a (possibly ordered) ring or a field.

For a set of indeterminates $\bar{X}$ we denote by $\mathcal{R}[\bar{X}]$ the polynomial ring over $\mathcal{R}$.

A **graph parameter** p is a function from the class of all finite graphs $Graphs$ into $\mathcal{R}$ which is invariant under graph isomorphism.

A **graph polynomial** p is a function from the class of all finite graphs $Graphs$ into $\mathcal{R}[\bar{X}]$ which is invariant under graph isomorphism.

Remark: In most situations in the literature $\mathcal{R}$ is $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$. The choice of the underlying ring or field may depend on the way we want to represent the graph parameter or graph polynomial.

For the graph parameter $d_{max}(G)$, the maximal degree of its vertices, $\mathbb{Z}$ suffices, but for $d_{average}(G)$, the average degree of its vertices, $\mathbb{Q}$ is needed.

Equivalence of graph polynomials, I

Let $\mathcal{C}$ be a graph property.
Let $P(G, \bar{X})$ and Let $Q(G, \bar{Y})$ be two graph polynomials.

Definition 23
We say that Q determines (induces) P over $\mathcal{C}$, or
Q is at least as distinctive than P over $\mathcal{C}$, and write $P \preceq^{\mathcal{C}}_{d.p.} Q$
if for all graphs G_1 and G_2 in $\mathcal{C}$,

$$Q(G_1) = Q(G_2) \textit{ implies that } P(G_1) = Q(P_2).$$

- If $\mathcal{C}$ consists of all graphs, we omit $\mathcal{C}$.
- The definition also applies to graph parameters $P(G), Q(G) \in \mathbb{Z}$.

P and Q are d.p.-equivalent over $\mathcal{C}$, and write $P \sim^{\mathcal{C}}_{d.p.} Q$,
iff $P \preceq^{\mathcal{C}}_{d.p.} Q$ and $Q \preceq^{\mathcal{C}}_{d.p.} P$

Examples of $P \preceq^{\mathcal{C}}_{d.p.} Q$

(i) (DKT, 3.2.1) The chromatic polynomial $\chi(G, X)$ determines the graph parameters $\mid V(G) \mid$, $\mid E(G) \mid$, $\chi(G)$, $k(G)$, $b(G)$, $g(G)$, etc.

(ii) d_{max} and $d_{average}$ are d.p-incomparable.

(iii) The Tutte polynomial $T(G, X, Y)$ determines $\chi(G, X)$ on connected graphs, but not on all graphs.

(iv) Assume $P(G; X), Q(G; X), U(G, X)$ are three polynomials and $P(G, X) = U(G, X) \cdot Q(G, X)$.
Let $\mathcal{C}_U$ be a class of graphs such that for all $G_1, G_2 \mathcal{C}_U$ we have $U(G_1, X) = U(G_2 : X)$. Then $P \preceq^{\mathcal{C}_U}_{d.p.} Q$.

(v) Let $\mathcal{F}$ be the class of forests. For the characteristic polynomial $char(G, \lambda)$ and the matching polynomial $dm(G, \lambda)$ and we have

$$char \sim^{\mathcal{F}}_{d.p.} dm.$$

P-unique and P-equivalent graphs

Definition 24 *Let $P = P(G; \bar{X})$ a graph polynomial and $\mathcal{C}$ a class of graphs.*

(i) *Two graphs G_1 and G_2 are P-equivalent for $\mathcal{C}$ if $P(G; \bar{X}) = P(G_1; \bar{X})$.*

(ii) *A graph $G \in \mathcal{C}$ is P-unique for $\mathcal{C}$ if for any other graph $G_1 \in \mathcal{C}$ with $P(G; \bar{X}) = P(G_1; \bar{X})$ the graph G_1 is isomorphic to G.*

(iii) *P is complete for $\mathcal{C}$ if every graph $G \in \mathcal{C}$ is $P(G; \bar{X})$-unique for $\mathcal{C}$.*

If $\mathcal{C}$ consists of all graphs we omit $\mathcal{C}$.

Proposition 25 *Let P and Q be graph polynomials such that $P \preceq^{\mathcal{C}}_{d.p.} Q$.*

(i) *If G_1 and G_2 are Q-equivalent for $\mathcal{C}$ then they are also P-equivalent for $\mathcal{C}$.*

(ii) *If G is P-unique for $\mathcal{C}$ then G is Q-unique for $\mathcal{C}$.*

(iii) *If P is complete for $\mathcal{C}$ then Q is complete for $\mathcal{C}$.*

More examples of induced graph invariants

- Adjoint polynomials
- χ-equivalent graphs (from [DKT, chapter 5])
- The two matching polynomials
- T-unique graphs
- Almost complete graph invariants

Skip more examples 2

Adjoint polynomials

Let $P(G, \lambda)$ be a graph polynomial.
We denote by $\bar{G}$ the complement graph of G.

The adjoint polynomial $\bar{P}(G, \lambda)$ is the polynomial defined by

$$\bar{P}(G, \lambda) =_{def} P(\bar{G}, \lambda)$$

- **Exercise:** $P \preceq^{\mathcal{C}}_{d.p.} \bar{P}$ iff $\bar{P} \preceq^{\mathcal{C}}_{d.p.} P$
- For the Tutte polynomial $T(G, X, Y)$ and $\bar{E}_n = K_n$ we have
  (i) $T(E_m) = T(E_n) = 1$ for all $n \in \mathbb{N}$.
  (ii) $T(K_m) \neq T(K_n)$ for $m \neq n$.
  (iii) Hence the Tutte polynomial and its adjoint are not d.p.-comparable.

Back to more example list 2

χ-equivalent graphs (from [DKT, chapter 5]), I

(i) The graphs E_n, K_n and $K_{n,n}$ are χ-unique for $n \geq 1$.

(ii) The graphs C_n are χ-unique for $n \geq 3$, $C_i = K_i$ for $i \leq 2$.

(iii) Any two trees on n vertices are χ-equivalent.

In [DKT, chapter 5] many pairs of χ-equivalent graphs are constructed using a method due to R.C. Read (1987) and G.L. Chia (1988).

Research project:

Study P-equivalence for the various generalized colorings of Lecture 10.

$char$-equivalent graphs, II

From M. Noy, Graphs determined by polynomial invariants (2003)

Let $char(G, x) = \det(x \cdot \mathbf{1} - A_G)$ be the characteristic polynomial of G with adjacency matrix A_G.

(i) The graphs $K_{n,n}$ are $char$-unique.

(ii) The line graphs $L(K_n)$ are $char$-unique for $n \neq 8$.
For $n = 8$ there are three exceptions.

(iii) The line graphs $L(K_{n,n})$ are $char$-unique for $n \neq 4$.
For $n = 4$ there is one exception.

Back to example list 2

The two matching polynomials

Recall, for $G = (V, E)$ with $\mid V \mid = n$,

$$dm(G, x) = \sum_r (-1)^r m_r(G) x^{n-2r}$$

be the (defect) matching polynomial and

$$gm(G, x) = \sum_r m_r(G) x^r$$

the (generating) matching polynomial.

We have

$$dm(G; x) = x^n gm(G; (-x)^{-2})$$

Graphs equivalent for matching polynomials.

From M. Noy, Graphs determined by polynomial invariants (2003)

- For every graph G we have $gm(G, x) = gm(G \sqcup E_n, x)$
  but $dm(G, x) \neq dm(G \sqcup E_n, x)$.

  $dm(P_2, x) = x^2 - 1$ and $dm(P_2 \sqcup E_k, x) = x^3 - x$,
  but $gm(P_2, x) = x^2 - 1$ and $gm(P_2 \sqcup E_k, x) = x^2 - 1$

- $| V(G) | \preceq_{d.p.} dm$, and therefore $gm \preceq_{d.p.} dm$.
  In other words gm is strictly less expressive than dm.

- $gm \sim_{d.p.} dm$ on graphs of a fixed number of vertices.

- The graphs $K_{n,n}$ are dm-unique.

  Are they also gm-unique?

Research project:

Study dm-equivalence and gm-equivalence of graphs further.

Back to example list 2

T-unique graphs

From A. de Mier and M. Noy, On Graphs determined by the Tutte polynomial (2004)

For a graph $G = (V, E)$ and $A \subseteq E$ we denote by $G[A] = (V, A)$ the spanning subgraph generated by A. We set $r(A) =| V | -k(G[A])$ and $n(A) =| A | -r(A)$.

The Tutte polynomial is defined by

$$T(G; X, Y) = \sum_{A \subseteq E} (X-1)^{r(E)-r(A)}(Y-1)^{n(A)}$$

(i) Recall that $\chi \preceq_{d.p.} T$ on **connected** graphs.
Hence the graphs $K_{n,n}$ are T-unique.

(ii) The wheels W_n are T-unique for all $n \in \mathbb{N}$.

Wheels are χ-unique for W_{2n}, W_5 and W_7 are not. In general it is not known (?) whether W_{2n+1} is χ-unique.

(iii) The ladders L_n are T-unique for all $n \geq 3$.

They are only known to be χ-unique for small values of n.

Bollobas-Pebody-Riordan Conjecture:

Almost all graphs are T-unique and even χ-unique

Let us make it more precise:

Let TU (χU) be the graph property:
$G \in TU$ $(G \in \chi U)$ iff G is T-unique (χ-unique),
and $TU(n)$ $(\chi U(n))$ be the density function of TU (χU).

The conjecture for the Tutte polynomial now is

$$\lim_{n \to \infty} \frac{TU(n)}{2^{\binom{n}{2}}} = 1$$

Similar for $\chi(G, \lambda)$.

Is TU (χU) definable in some logic with a $0-1$-law?

B. Bollobás, L. Pebody and O. Riordan, Contraction-Deletion Invariants for Graphs,

Journal of Combinatorial Theory, Serie B, vol. 80 (2000) pp. 320-345.

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Almost complete graph invariants

A graph polynomial P is almost complete, if almost all graphs are P-unique.

Research problems:

- Study the definability of the graph property G is P unique for various graph polynomials P.
- Find natural graph polynomials which are almost complete.
- In particular, is the signed Tutte polynomial T_{signed} almost complete for signed graphs.

  A positive answer would be interesting for knot theorists: T_{signed} is intimately related to the Jones polynomial of knot theory.

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Comparison of graph polynomials by coefficients

Coefficients of graph polynomials, I: The univariate case

We denote by $\mathbb{Z}^{<\omega}$ the finite sequences of elements of $\mathbb{Z}$.

Let $P(G,X) \in \mathbb{Z}[X]$ and $P(G,X) = \sum_{i=0}^{d(G)} a_i(G) \cdot X^i$ with $a(G)_{d(G)} \neq 0$.

We denote by $cP(G,X)$ the finite sequence $(a_i(G))_{i \leq d(G)} \in \mathbb{Z}^{<\omega}$.

$cP(G,X)$ are the (standard) coefficients of $P(G,X)$, and $d(G)$ is its degree.

c is a one-one and onto function $c : \mathbb{Z}[X] \xrightarrow{c} \mathbb{Z}^{<\omega}$.

Instead of looking at graph polynomials $P : Graphs \xrightarrow{P} \mathbb{Z}[X]$, we can look at the function $cP : Graphs \longrightarrow \mathbb{Z}^{<\omega}$ defined by

$$cP : Graphs \xrightarrow{P} \mathbb{Z}[X] \xrightarrow{c} \mathbb{Z}^{<\omega}$$

Lemma 26
For all graphs G_1, G_2, we have that $P(G_1) = P(G_2)$ iff $cP(G_1) = cP(G_2)$.

Other representations of graph polynomials

Our definition of cP uses the **power form of P**.

We could have used also factorial form or binomial form of P.

- cP denotes the coefficients of P in power form.
- c_1P denotes the coefficients of P in factorial form.
- c_2P denotes the coefficients of P in binomial form.

We note that there are simple algorithms to pass from one representation to another.

Equivalence of graph polynomials, II

Let $\mathcal{C}$ be a graph property.
Let $P(G, \bar{X})$ and Let $Q(G, \bar{Y})$ be two graph polynomials.

Definition 27
We say that Q determines coefficient-wise P over $\mathcal{C}$ and write $P \preceq^{\mathcal{C}}_{coeff} Q$ if there is a function $F : \mathbb{Z}^{<\omega} \to \mathbb{Z}^{<\omega}$ such that for all graphs $G \in \mathcal{C}$

$$F(cQ(G)) = cP(G)$$

P and Q are coefficient-equivalent over $\mathcal{C}$, and write $P \sim^{\mathcal{C}}_{coeff} Q$, iff $P \preceq^{\mathcal{C}}_{coeff} Q$ and $Q \preceq^{\mathcal{C}}_{coeff} P$

- If $\mathcal{C}$ consists of all graphs, we omit $\mathcal{C}$.
- The definition also applies to graph parameters $P(G), Q(G) \in \mathbb{Z}$.
- Our definition is invariant under the choice of representations cP, c_1P or c_2P.

An example: F can be arbitrarily complex

Let $P(G, \lambda) = \sum_i a_i(G)\lambda^i$.

Let $P_{exp}(G, \lambda) = \sum_i 2^{a_i(G)}\lambda^i$,

and for $g : \mathbb{N} \to \mathbb{N}$ one-one and onto let $P_g(G, \lambda) = \sum_i a_i(G)\lambda^{g(i)}$.

Clearly,

$$P \sim_{coeff} P_g \sim_{coeff} P_{exp}$$

- If g is not computable, then F showing that $P \sim_{coeff} P_g$ cannot be computable in the **Turing model** of computation.
- Furthermore, F showing that $P \sim_{coeff} P_{exp}$ cannot be computable in the **Blum-Shub-Smale model** of computation.

Theorem 28 $P \preceq^{\mathcal{C}}_{coeff} Q$ *iff* $P \preceq^{\mathcal{C}}_{d.p.} Q$

Proof: $P \preceq^{\mathcal{C}}_{coeff} Q$ implies $P \preceq^{\mathcal{C}}_{d.p.} Q$.

Assume there is a function $F : \mathbb{Z}^{<\omega} \to \mathbb{Z}^{<\omega}$ such that for all graphs $G \in \mathcal{C}$ we have $F(cQ(G)) = cP(G)$.

Now let $G_1, G_2 \in \mathcal{C}$ such that $Q(G_1) = Q(G_2)$.

By Lemma 26 we have $cQ(G_1) = cQ(G_2)$.

Hence $F(cQ(G_1)) = F(cQ(G_2))$.

Since for all $G \in \mathcal{C}$ we have $F(cQ(G)) = cP(G)$, we get $cP(G_1) = cP(G_2)$ and, using Lemma 26 again, we have $P(G_1) = P(G_2)$.

Proof: $P \preceq^{\mathcal{C}}_{d.p} Q$ implies $P \preceq^{\mathcal{C}}_{coeff} Q$.

We use the well-ordering principle which equivalent to axiom of choice.

Let $\{F_\alpha : \alpha < \beta\}$ be a well-ordering of all the functions $F : \mathbb{Z}^{<\omega} \to \mathbb{Z}^{<\omega}$.

For $G \in \mathcal{C}$, let $\gamma(G) < \beta$ be the smallest ordinal such that $F_{\gamma(G)}(cQ(G)) = cP(G)$.

Now given $P(G, X) \preceq_{d.p.} Q(G, X)$, we define a function $F_{P,Q} : \mathbb{Z}^{<\omega} \to \mathbb{Z}^{<\omega}$ as follows:

$$F_{P,Q}(cQ(G)) = \begin{cases} F_{\gamma(G)}(cQ(G)) & \text{if } G \in \mathcal{C} \\ 0 & \text{else} \end{cases}$$

Using Lemma 26 and $P(G, X) \preceq_{d.p.} Q(G, X)$, this indeed defines a function.

Finally, as $F_{\gamma(G)}(cQ(G)) = F_{\gamma(G)}(cP(G))$, we get

$$F_{P,Q}(cQ(G)) = cP(G)$$

Q.E.D.

A proof without well-ordering (suggested by Ofer David)

Let S be a set of finite graphs and $s \in \mathbb{Z}^{<\omega}$.
For a graph polynomial P we define:

$P[S] = \{s \in \mathbb{Z}^{<\omega} : cP(G) = s$ for some $G \in S\}$ and $P^{-1}(s) = \{G : cP(G) = s\}$.

Now assume $P(G, X) \preceq_{d.p.} Q(G, X)$.

If $Q^{-1}(s) \neq \emptyset$, then for every $G_1, G_2 \in Q^{-1}(s)$ we have $cQ(G_1) = cQ(G_2)$, and therefore $cP(G_1) = cP(G_2)$.

Hence $P[Q^{-1}(s)] = \{t_s\}$ for some $t_s \in \mathbb{Z}^{<\omega}$.

Now we define

$$F_{P,Q}(s) = \begin{cases} t_s & Q^{-1}(s) \neq \emptyset \\ s & \text{else} \end{cases}$$

Q.E.D.

Example, I: The two matching polynomials

$dm(G, x) = \sum_r (-1)^r m_r(G) x^{n-2r}$

$gm(G, x) = \sum_r m_r(G) x^r$

We have $dm(G; x) = x^n gm(G; (-x)^{-2})$ where $n =| V |$.

- The degree of dm is n
- If $m_r(G) \neq 0$ the $n - 2r > 0$.
- Hence
$$\frac{dm(G; x)}{X^n}$$
is a polynomial, and we can compute the coefficients of gm from the coefficents of dm.
- We cannot compute the coefficients of dm from gm without knowing the value of $| V |= n$.

Example II: The Tutte polynomial and the chromatic polynomial

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The Tutte polynomial and the chromatic polynomial are related by the formula

$$\chi(G, X) = (-1)^{r(G)} \cdot X^{k(G)} \cdot T(G; 1 - X, 0)$$

- To compute the coefficients of $\chi(G; X)$ from $T(G; X, Y)$ we have to know the parity of $r(G)$ and the number of connected components of G.
- For connected graphs $k(G) = 1$ and $r(G) = \mid V \mid -1$.

Introducing auxiliary parameters $\mathcal{S}$

Let $\mathcal{S} = \{S_1(G), \ldots, S_t(G)\}$ be graph parameters (polynomials), and $\mathcal{C}$ a graph property.
Let $P(G, \bar{X})$ and Let $Q(G, \bar{Y})$ be two graph polynomials.

Definition 29
We say that Q determines P relative to $\mathcal{S}$ over $\mathcal{C}$, or
Q is at least as distinctive than P relative to $\mathcal{S}$ over $\mathcal{C}$, and write $P \preceq^{\mathcal{S},\mathcal{C}}_{r.d.p.} Q$
if for all graphs $G_1, G_2 \in \mathcal{C}$ with $S_i(G_1) = S_i(G_2) : i \leq t$ we have

$$Q(G_1) = Q(G_2) \text{ implies that } P(G_1) = Q(P_2).$$

Definition 30
We say that Q determines coefficient-wise P relative to $\mathcal{S}$ over (C)
and write $P \preceq^{\mathcal{S},(C)}_{relcoeff} Q$
if there is a function $F : (\mathbb{Z}^{<\omega})^{t+1} \rightarrow \mathbb{Z}^{<\omega}$ such that for all graphs $G \in \mathcal{P}$

$$F(cS_1(G), \ldots, cS_t(G), cQ(G)) = cP(G)$$

The equivalence relations $P \sim^{\mathcal{S},(C)}_{r.d.p.} Q$ and $P \sim^{\mathcal{S},(C)}_{relcoeff} Q$, are defined as usual.

Theorem 31 $P \preceq^{\mathcal{S}}_{relcoeff} Q$ *iff* $P \preceq^{\mathcal{S}}_{r.d.p.} Q$

The proof is left as an exercise!

Back to outline

Semantic Properties of Graph Polynomials

Similar graphs and similarity functions

Two graphs G_1, G_2 are similar if they have the same number of vertices, edges and connected components, i.e.,

- $|V(G_1)| = n(G_1) = n(G_2) = |V(G_2)|$,
- $|E(G_1)| = m(G_1) = m(G_2) = |E(G_2)|$, and
- $k(G_1) = k(G_2)$.
- $\mathcal{S} = \{|V(G)|, |E(G)|, k(G)\}$

A graph parameter or graph polynomial is a similarity function if it is invariant and similarity.

(i) The nullity $\nu(G) = m(G) - n(G) + k(G)$ and the rank $\rho(G) = n(G) - k(G)$ of a graph G are similarity polynomials with integer coefficients.

(ii) Similarity polynomials can be formed inductively starting with similarity functions $f(G)$ not involving indeterminates, and monomials of the form $X^{g(G)}$ where X is an indeterminate and $g(G)$ is a similarity function not involving indeterminates. One then closes under pointwise addition, subtraction, multiplication and substitution of indeterminates X by similarity polynomials.

Comparing graph polynomials up to graph similarity

In the literature graph polynomials are mostly compared up to graph similarity:

- We note that the various matching polynomials are not d.p.-equivalent. The number of vertices of a graph G is not induced by all its variations, but is induced by some of them.

- However, if restricted to similar graphs, all the matching polynomials have the same distinctive power.

- Similarily, the Tutte polynomial does not induce the chromatic polynomial. They behave differently on empty graphs. However, on similar graphs, the Tutte polynomial determines the chromatic polynomial.

This leads to the following definitions.

Distinctive power of graph polynomials over $\mathcal{S}$, I

Two graph polynomials are usually compared via their distinctive power.

A graph polynomial $Q(G,X)$ is less s-distinctive than $P(G,Y)$, $Q \preceq_s P$, if for every two similar graphs G_1 and G_2

$$P(G_1,X) = P(G_2,X) \text{ implies } Q(G_1,Y) = Q(G_2,Y).$$

We also say the $P(G;X)$ s-determines $Q(G;X)$ if $Q \preceq P$.

Two graph polynomials $P(G,X)$ and $Q(G,Y)$ are s-equivalent in distinctive power (s.d.p-equivalent) if for every two similar graphs G_1 and G_2

$$P(G_1,X) = P(G_2,X) \text{ iff } Q(G_1,Y) = Q(G_2,Y).$$

The same definition also works for graph parameters and multivariate graph polynomials.

Distinctive power of graph polynomials relative to $\mathcal{S}$, II

Let $\mathcal{R}$ be the ring of coefficients of our graph polynomials, and let $\mathcal{R}^\infty$ denotes the set of finite sequences of $\mathcal{R}$.
We denote by $cP(G) \in \mathcal{R}^\infty$ the sequence of coefficients of $P(G, X)$.

Proposition 32
Two graph polynomials $P(G, X_1, \ldots X_r)$ and $Q(G, Y_1, \ldots, Y_s)$ are s-equivalent in distinctive power (s.d.p-equivalent) over $\mathcal{S}$ *($P \sim_{d.p.} Q$) iff there are two functions $F_1, F_2 : \mathcal{R}^\infty \to \mathcal{R}^\infty$ such that for every graph G*

$$F_1(n(G), m(G), k(G), cP(G)) = cQ(G) \text{ and}$$
$$F_2(n(G), m(G), k(G), cQ(G)) = cP(G)$$

Proposition 32 shows that our definition of equivalence of graph polynomials is mathematically equivalent to the definition proposed by
C. Merino and S. Noble in 2009.

Computability

The functions F_1, F_2 in Proposition 32 need not be computable in any sense, even if the coefficients of $P(G)$ and $Q(G)$ are integers.

A graph polynomial $P(G; X)$ with coefficients in a ring $\mathcal{R}$ is computable (in a suitable model of computation for $\mathcal{R}$) if

(i) the function $cP : \mathcal{G} \rightarrow \bigcup_n \mathcal{R}^n$ computing the coefficients of $P(G; X)$ is computable, and

(ii) the decision problem, given $s \in \bigcup_n \mathcal{R}^n$ is there a graph with $cP_(G) = s$ is decidable.

Theorem 33
Let $P(G; X)$ and $Q(G; X)$ be two computable graph polynomials which are d.p.-equivalent. Then there are F_1, F_2 as in Proposition 32 which are computable.

In this case we say that $P(G; X)$ and $Q(G; X)$ are computably d.p.-equivalent.

Prefactor and subtsitution equivalence, I

- We say that $P(G;\bar{X})$ is prefactor reducible to $Q(G;\bar{X})$ and we write
$$P(G;\bar{Y}) \preceq_{prefactor} Q(G;\bar{X})$$
if there are similarity functions
$$f(G;\bar{X}), g_1(G;\bar{X}), \ldots, g_r(G;\bar{X})$$
such that
$$P(G;\bar{Y}) = f(G;\bar{X}) \cdot Q(G; g_1(G;\bar{Y}), \ldots, g(G;\bar{Y})).$$

- We say that $P(G;\bar{X})$ is substitutions reducible to $Q(G;\bar{X})$, and we write
$$P(G;\bar{Y}) \preceq_{subst} Q(G;\bar{X})$$
if $P(G;\bar{Y}) \preceq_{prefactor} Q(G;\bar{X})$ and, additionally, $f(G;\bar{X}) = 1$ for all values of $\bar{X}$.

- $P(G;\bar{X})$ and $Q(G;\bar{X})$ are prefactor (substitution) equivalent if the relationship holds in both directions.

It follows that if $P(G; \bar{X})$ and $Q(G; \bar{X})$ are prefactor (substitution) equivalent then they are computably d.p.-equivalent.

Semantic properties of graph parameters

A semantic property (s-semantic property) is a class of graph parameters (polynomials) closed under d.p.-equivalence or (s.d.p.-equivalence).

Let $p(G)$ be a graph parameter with values in $\mathbb{N}$, and $P(G;X)$ be a graph polynomial.

- The degree of $P(G;X)$ equals $p(G)$ is not a semantic property of $P(G;X)$.

  Using Proposition 32 we see that $P(G;X)$ and $P(G;X^2)$ are d.p.-equivalent, but they have different degrees.

- $P(G;X)$ determines $p(G)$ is a semantic property of $P(G;X)$.

  The degree of $P(G;X)$ equals $p(G)$ is an accidental result of the particular presentation of $P(G;X)$.

- The number of triangles of G is determined by the characteristic polynomial, but that it is twice the absolute value of the third coefficient again is a result of its particular presentation.

Semantic vs syntactic properties of graph polynomials, I

Semantically meaningless properties:

(i) $P(G, X)$ is monic for each graph G, i.e., the leading coefficient of $P(G; X)$ equals 1.

Multiplying each coefficient by a fixed polynomial gives an equivalent graph polynomial.

(ii) The leading coefficient of $P(G, X)$ equals the number of vertices of G.

However, proving that two graphs G_1, G_2 with $P(G_1, X) = P(G_2, X)$ have the same number of vertices is semantically meaningful.

(iii) The graph polynomials $P(G; X)$ and $Q(G; X)$ coincide on a class $\mathcal{C}$ of graphs, i.e. for all $G \in \mathcal{C}$ we have $P(G; X) = Q(G; X)$.

The semantic content of this situation says that if we restrict our graphs to $\mathcal{C}$, then $P(G; X)$ and $Q(G; X)$ have the same distinguishing power.

The equality of $P(G; X)$ and $Q(G; a)X$ is a syntactic conincidence or reflects a clever choice in the definitions $P(G; X)$ and $Q(G; X$.

Semantic vs syntactic properties of graph polynomials, II

Clever choices of can be often achieved.

Let $\mathcal{C}$ be class of finite graphs closed under graph isomorphisms.

Proposition 34
Assume that $P(G;X)$ and $Q(G;X)$ have the same distinguishing power on a class of graphs $\mathcal{C}$. Then there is $P' \sim_{d.p.} P$ such that the graph polynomials $P'(G;X)$ and $Q(G;X)$ coincide on a class $\mathcal{C}$ of graphs.

If, additionally, $\mathcal{C}, P(G;X)$ and $Q(G;X)$ are computable, then $P'(G;X)$ can be made computable, too.

Proposition 34 also holds when we replace computable by definable in SOL, as we shall see later.

Thank you for your attention

Back to outline 1 of Lecture 1

Back to outline 1 of Prague Lectures

Outline of the course

LECTURE 00: Second Order Logic (SOL) and its fragments (Background, not lectured)
LOGICS (14 slides)

LECTURE 01: Friday, Oct 10, 2014, 14:00-15:40, Prague Lecture 1,
A landscape of graph parameters and graph polynomials. Comparing graph parameters. Towards a general theory.
(90 minutes, 90 slides with skip-options)

LECTURE 02: Thursday, Oct 16, 2014, 12:20-14:00 Prague Lecture 2,
Why is the chromatic polynomial a polynomial? Where to graph polynomial occur naturally? Definability of graph properties and graph polynomials in fragment of Second Order Logic.
(90 minutes, ca. 99 slides with skip options)

LECTURE 03: Thursday, Oct 16, 2014, 14:30-16:00 Prague Lecture 3,
Connection matrices for graph parameters. When do connection matrices of graph parameters have finite rank? Connection matrices for graph parameters definable in fragments of Second Order Logic. The finite rank theorem. Using connections matrices to prove non-definability.
(90 minutes, ca. 55 slides with skip options)

Further links to the literature.

Further links

[arXiv] J.A. Makowsky's Graph Polynomial Go to **Homepage** at http://www.cs.technion.ac.il/~janos/RESEARCH/gp-homepage.html

[KMR 2013] J. A. Makowsky T. Kotek and E. V. Ravve, **A computational framework for the study of partition functions and graph polynomials**. In Proceedings of the 12th Asian Logic Conference ?11, pages 210?230, 2013. download at http://www.cs.technion.ac.il/~janos/RESEARCH/alcpaper.pdf

[GKM 2012] B. Godlin, E. Katz and J. A. Makowsky, **Graph Polynomials: From Recursive Definitions to Subset Expansion Formulas**. J. Log. Comput. 22(2): 237-265 (2012) download at http://www.cs.technion.ac.il/~janos/RESEARCH/GodlinKatzMakowsky.pdf

[M 2008] J.A. Makowsky, **From a Zoo to a Zoology: Towards a general theory of graph polynomials**, Theory of Computing Systems, 2008. download at http://dx.doi.org/10.1007/s00224-007-9022-9

More links

Further links, II

[arXiv] J.A. Makowsky's papers at http://arxiv.org/find/all/1/au:+Makowsky/0/1/0/all/0/1?per_page=100 on **arXiv**.

[dblp] J.A. Makowsky's papers at http://www.informatik.uni-trier.de/~ley/pers/hd/m/Makowsky:Johann_A=.html on **DBLP**.

[google] J.A. Makowsky's papers at http://scholar.google.co.il/citations?hl=en&user=ooNKL6UAAAAJ&pagesize=100&view_op=list_works at **scholar.google**.

[Course notes] J.A. Makowsky's Course notes.

[PhD Theses] PhD Theses on graph polynomials (a selection)

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Further links: Course notes

Slides of courses on graph polynomials and related topics:

Technion 2005/6 Lecture notes of **Advanced Topics in Computer Science (238900)**

Technion 2009/10 Lecture notes of **Advanced Topics in Computer Science (236605)**

Vienna 2014 Lecture notes of **EMCL Lecture 2014: Graph polynomials**

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Further links: PhD Theses

PhD Theses on graph polynomials and related topics:

I. Averbuch PhD Thesis (Technion 2011): **Completeness and Universality Properties of Graph Invariants and Graph Polynomials**

T. Kotek PhD Thesis (Technion 2012): **Definability of combinatorial functions**

M. Trinks PhD Thesis (TU Freiberg 2012): **Graph Polynomials and Their Representations**

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