Lecture 2

Why is the chromatic polynomial a polynomial?

Taming the class of graph polynomials

Definability of graph properties and
numeric graph parameters

Outline of Prague Lecture 2

- Variations of the chromatic polynomial
- Why are there many chromatic polynimals?
- The classical graph polynomials
- The matching polynomials: A case study
- Second Order Logic (SOL)
- Graph properties
- Logic and Complexity
- HEX and Variations
- The role of order
- Definability of numeric graph invariants

Thanks

The Chromatic Polynomial

and

Its Variations

Back to outline of lecture 2

The (vertex) chromatic polynomial

Let $G = (V(G), E(G))$ be a graph, and $\lambda \in \mathbb{N}$.

A **λ-vertex-coloring** is a map

$$c : V(G) \to [\lambda]$$

such that $(u, v) \in E(G)$ implies that $c(u) \neq c(v)$.

We define $\chi(G, \lambda)$ to be the number of λ-vertex-colorings

Theorem: (G. Birkhoff, 1912)

$\chi(G, \lambda)$ is a polynomial in $\mathbb{Z}[\lambda]$.

Proof:

(i) $\chi(E_n) = \lambda^n$ where E_n consists of n isolated vertices.

(ii) For any edge $e = E(G)$ we have $\chi(G - e, \lambda) = \chi(G, \lambda) + \chi(G/e, \lambda)$.

Interpretation of $\chi(G, \lambda)$ for $\lambda \notin \mathbb{N}$

What's the point in considering $\lambda \notin \mathbb{N}$?

Stanley, 1973 For simple graphs G, $|\,\chi(G, -1)\,|$ counts the number of acyclic orientations of G.

Stanley, 1973 There are also combinatorial interpretations of $\chi(G, -m)$ for each $m \in \mathbb{N}$, which are more complicated to state.

Open: What about $\chi(G, \lambda)$ for each $m \in \mathbb{R} - \mathbb{Z}$?

The Four Color Conjecture

Birkhoff wanted to prove the Four Color Conjecture using techniques from real or complex analysis.

Conjecture:(Birkhoff and Lewis) If G is planar then $\chi(G,\lambda) \neq 0$ for $\lambda \in [4, +\infty) \subseteq \mathbb{R}$.

This was not very successful. However, for real roots of χ we know:

Jackson, 1993 For simple graphs G we have $\chi(G,\lambda) \neq 0$ for $\lambda \in (-\infty, 0)$, $\lambda \in (0,1)$ and $\lambda \in (1, \frac{32}{27})$.

Birkhoff and Lewis, 1946 For planar graphs G we have $\chi(G,\lambda) \neq 0$ for $\lambda \in [5, +\infty)$.

Still open: Are there planar graphs G such that $\chi(G,\lambda) = 0$ for some $\lambda \in (4,5)$?

Thomassen, 1997 and Sokal, 2004 The real roots of all chromatic polynomials are dense in $(\frac{32}{27}, k]$ for graphs of tree-width at most k. The complex roots are dense in $\mathbb{C}$.

Variations on coloring, I

We can count other coloring functions.

- proper λ-edge-colorings:

  $f_E : E(G) \rightarrow [\lambda]$ such that if $(e, f) \in E(G)$ have a common vertex then $f_E(e) \neq f_E(f)$.

  $\chi_e(G, \lambda)$ denotes the number of λ- edge-colorings

- Total colorings

  $f_V : V \rightarrow [\lambda_V]$, $f_E : E \rightarrow [\lambda_E]$ and $f = f_V \cup f_E$,
  with f_V a proper vertex coloring and f_E a proper edge coloring.

- Connected components

  $f_V : V \rightarrow [\lambda_V]$, If $(u, v) \in E$ then $f_V(u) = f_V(v)$.

- Hypergraph colorngs

  Vitaly I. Voloshin, Coloring Mixed Hypergraphs: Theory, Algorithms and Applications, Fields Institute Monographs, AMS 2002

Fact: The corresponding counting functions are polynomials in λ.

Variations on coloring, II

Let $f : V(G) \to [\lambda]$ be a function, such that Φ is one of the properties below and $\chi_\Phi(G, \lambda)$ denotes the number of such colorings with atmost λ colors.

* **convex:** Every monochromatic set induces a connected graph.

* **injective:** f is injectiv on the neighborhood of every vertex.

- **complete:** f is a proper coloring such that every pair of colors occurs along some edge.

* **harmonious:** f is a proper coloring such that every pair of colors occurs at most once along some edge.

- **equitable:** All color classes have (almost) the same size.

* **equitable, modified:** All non-empty color classes have the same size.

Fact: For (*), $\chi_\Phi(G, \lambda)$ is a polynomial in λ, for (-), it is not.

Variations on coloring, III

* **path-rainbow:** Let $f : E \to [\lambda]$ be an edge-coloring. f is **path-rainbow** if between any two vertices $u, v \in V$ there as a path where all the edges have different colors.

Fact: $\chi_{rainbow}(G, \lambda)$, the number of path-rainbow colorings of G with λ colors, is a polynomial in λ

Rainbow colorings of various kinds arise in computational biology

* **-monochromatic components:** Let $f : V \to [\lambda]$ be an vertex-coloring and $t \in \mathbb{N}$. f is an mcc_t-coloring of G with λ colors, if all the connected components of a monochromatic set have size at most t.

Fact: For fixed $t \geq 1$ the function $\chi_{mcc_t}(G, \lambda)$, the number of mcc_t-colorings of G with λ colors, is a polynomial in λ. but not in t.

mcc_t colorings were first studied in:

N. Alon, G. Ding, B. Oporowski, and D. Vertigan. Partitioning into graphs with only small components. *Journal of Combinatorial Theory, Series B*, 87:231–243, 2003.

Variations on coloring, IV

Let $\mathcal{P}$ be any graphs property and let $n \in \mathbb{N}$.

We can define coloring functions $f : V \to [\lambda]$ by requiring that the union of any n color classes induces a graph in $\mathcal{P}$.

- For $n = 1$ and $\mathcal{P}$ the empty graphs $G = (V, \emptyset)$ we get the proper colorings.
- For $n = 1$ and $\mathcal{P}$ the connected graphs we get the convex colorings.
- For $n = 1$ and $\mathcal{P}$ the graphs which are disjoint unions of graphs of size at most t, we get the mcc_t-colorings.
- For $n = 2$ and $\mathcal{P}$ the acyclic graphs we get the **acyclic colorings**, introduced in: B. Grunbaum, Acyclic colorings of planar graphs, Israel J. Math. 14 (1973), 390-412

Theorem: Let $\chi_{\mathcal{P},n}(G, \lambda)$ be the number of colorings of G with λ colors such that the union of any n color classes induces a graph in $\mathcal{P}$.

Then $\chi_{\mathcal{P},n}(G, \lambda)$ is a polynomial in λ.

Variations on colorings, V: coloring relations

Let $G = (V, E)$. Here we look at an example where the coloring is a relation $R \subseteq V \times [k]$ rather than a function $f : V \to [k]$.
We denote by C_v the set $\{c \in [k] : (v, c) \in R\}$.

Let $a, b \in \mathbb{N}$. An (a, b)-coloring relation with k colors is a relation $R \subseteq V \times [k]$ such that

- For each $v \in V$ there are at most a-many colors $c \in [k]$ such that $(v, c) \in R$.
- If $(u, v) \in E$ then $C_u \neq C_v$ and there are at most b-many distinct elements $c_1, \ldots, c_b$ in $C_u \cap C_v$.

Exercise:

- Compute the number of (a, b)-coloring relations of the complete graphs K_n for various $a, b, k \in \mathbb{N}$.
- Is the number (a, b)-coloring relations with k colors of a graph G a polynomial in a, b or k?
- Look at the corresponding definitions with "at most" replaced by "at least" or "exactly".

Variations on colorings, VI: Two kinds of colors.

Let $G = (V, E)$.
Here we look at two disjoint color sets $A = [k_1]$ and $B = [k_1 + k_2] - [k_1]$.
The colors in A are called proper colorings.
Our coloring is a function $f : V \to [k_1 + k_2] = [k]$ such that

- If $(u, v) \in E$ and $f(u) \in A$ and $f(v) \in A$ then $f(u) \neq f(v)$.

- We count the number of colorings with $k = k_1 + k_2$ colors such that k_1 colors are in A, i.e., proper.

Theorem 1 (K. Dohmen, A. Pönitz and P. Tittman, 2003)
This gives us a polynomial $P(G, k_1, k)$ in k_1 and k.

Coloring Properties

Why are there many chromatic polynomials?

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Colorings

Our framework is as follows:

- Let $\mathfrak{M}$ be a finite relational τ-structure with universe M.
- Let $k \in \mathbb{N}$ and $[k] = \{0, \ldots, k-1\}$.
- Let f be an r-ary function $f : M^r \to [k]$.
- We shall look at families $\mathcal{P}$ consisting of triples of the form $(\mathfrak{M}, f, [k])$.

Coloring properties, I

A class of such triples $\mathcal{P}$ is a **coloring property** if

Extension Property: Let $n \leq k_i, i = 1, 2$ and let
$(\mathfrak{M}, f, [k_1])$ and $(\mathfrak{M}, f, [k_2])$ be two colorings of $\mathfrak{M}$,
using only colors in $[n]$, i.e., $rg(f) \subseteq [n]$.
Then $(\mathfrak{M}, f, [k_1]) \in \mathcal{P}$ iff $(\mathfrak{M}, f, [k_2]) \in \mathcal{P}$.

Isomorphism Property: $\mathcal{P}$ is closed under isomorphisms of colorings.

The isomorphisms property implies the permutation property:

Permutation Property: Let $f : M^r \rightarrow [k]$ be a fixed coloring.
For π is a permutation of $[k]$, we define the coloring f_π by $f_\pi(\bar{a}) = \pi(f(\bar{a}.$
Then $(\mathfrak{M}_k, f, [k]) \in \mathcal{P}$ iff $(\mathfrak{M}_k, f_\pi, [k]) \in \mathcal{P}$.

Coloring properties, II

If instead of coloring functions f we allow coloring relations

$$R \subseteq M^R \times [k]$$

we need some additional properties:

(i) A coloring property $\mathcal{P}$ of triples $(\mathfrak{M}_k, R, [k]) \in \mathcal{P}$ is **bounded**, if for every $\mathfrak{M}$ there is a number N_M such that for all $k \in \mathbb{N}$ the set of colors

$$\{x \in [k] : \exists \bar{y} \in M^r R(\bar{y}, x)\}$$

has size at most N_M.

(ii) A coloring property is **range bounded**, if its range is bounded in the following sense: There is a number $d \in \mathbb{N}$ such that for every $\mathcal{M}$ and $\bar{y} \in M^r$ the set $\{x \in [k] : R(\bar{y}, x)\}$ has at most d elements.

Clearly, if a coloring property is range bounded, it is also bounded.

Coloring properties, III

We denote by

$$\chi_{\mathcal{P}}(\mathfrak{M}, k)$$

the number of generalized $k - \mathcal{P}$-coloring R on $\mathfrak{M}$.

Tomer Kotek, Johann A. Makowsky, and Boris Zilber

On Counting Generalized Colorings

to appear in:

Model Theoretic Methods of Finite Combinatorics,

M. Grohe and J.A. Makowsky eds.

Contemprorary Mathematics Series of the AMS (2011)

Conference version in: Computer Science Logic, CSL'08, vol. 5213, (2008), 339-353

Uniform Definability in Logical Formalisms

Let ϕ be a sentence of some logic $\mathcal{L}$.

$\mathcal{L}$ could be first order logic FOL, second order logic SOL, monadic second order logic MSOL, or some fragment thereof.

We shall be interested in cases where the coloring property $\mathcal{P}$ is definable in $\mathcal{L}$ by a formula $\phi(P) \in \mathcal{L}$.

If $\phi(P)$ defines a (bounded) coloring property, we say that
$\phi(P)$ is a coloring formula.

If $\mathcal{P}$ is $\mathcal{L}$-definable we call $\chi_{\mathcal{P}}(\mathfrak{M}, k)$ an $\mathcal{L}$-chromatic counting function and write

$$\chi_{\phi(P)}(\mathfrak{M}, k) = \chi_{\mathcal{P}}(\mathfrak{M}, k).$$

All the examples encountered so far are SOL-chromatic counting function.

Generalized multi-colorings

To construct also graph polynomials in several variables, we extend the definition to deal with several color-sets, and also call them generalized chromatic polynomials.

We say an $(\alpha + 2)$-tuple

$$(\mathfrak{M}, R, [k_1], \ldots, [k_\alpha],)$$

with

$$R \subset M^m \times [k_1]^{m_1} \times \ldots \times [k_\alpha]^{m_\alpha}$$

is a generalized multi-coloring of $\mathfrak{M}$ for colors $\bar{k}^\alpha = (k_1, \ldots, k_\alpha)$.

The extension and isomorphism property are adapted appropriately to deal also with unused color-sets.

By abuse of notation, $m_i = 0$ is taken to mean the color-set k_i is not used in R.

A theorem with an elementary generic proof

THEOREM:

For every $\mathcal{M}$ the counting function $\chi_{\phi(R)}(\mathcal{M}, k)$ is a polynomial in k of the form

$$\sum_{j=0}^{d\cdot|M|^m} c_{\phi(R)}(\mathcal{M}, j)\binom{k}{j}$$

where $c_{\phi(R)}(\mathcal{M}, j)$ is the number of generalised $k-\phi$-colorings R with a fixed set of j colors.

Polynomials in $\mathbb{Z}[k]$ with monomials of the form $\binom{k}{j}$ are sometimes called **Newton polynomials**.

In the light of this theorem we call $\chi_{\phi(R)}(\mathcal{M}, k)$ also a

generalised chromatic polynomial.

Proof

We first observe that any generalised coloring R uses at most

$$N = d \cdot \mid M \mid^m$$

of the k colors.

For any $j \leq N$, let $c_{\phi(R)}(\mathcal{M}, j)$ be the number of colorings, with a fixed set of j colors, which are generalised vertex colorings and use all j of the colors.

Next we observe that any permutation of the set of colors used is also a coloring.

Therefore, given k colors, the number of vertex colorings that use exactly j of the k colors is the product of $c_{\phi(R)}(\mathcal{M}, j)$ and the binomial coefficient $\binom{k}{j}$. So

$$\chi_{\phi(R)}(\mathcal{M}, k) = \sum_{j \leq N} c_{\phi(R)}(\mathcal{M}, j) \binom{k}{j}$$

The right side here is a polynomial in k, because each of the binomial coefficients is. We also use that for $k \leq j$ we have $\binom{k}{j} = 0$. Q.E.D.

The Classical Graph Polynomials

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Prominent (classical) graph polynomials

- The chromatic polynomial (G. Birkhoff, 1912)
- The Tutte polynomial and its colored versions
  (W.T. Tutte 1954, B. Bollobas and O. Riordan, 1999);
- The characteristic polynomial
  (T.H. Wei 1952, L.M. Lihtenbaum 1956, L. Collatz and U. Sinogowitz 1957)
- The various matching polynomials (O.J. Heilman and E.J. Lieb, 1972)
- Various clique and independent set polynomials (I. Gutman and F. Harary 1983)
- The Farrel polynomials (E.J. Farrell, 1979)
- The cover polynomials for digraphs (F.R.K. Chung and R.L. Graham, 1995)
- The interlace-polynomials
  (M. Las Vergnas, 1983, R. Arratia, B. Bollobás and G. Sorkin, 2000)
- The various knot polynomials (of signed graphs)
  (Alexander polynomial, Jones polynomial, HOMFLY-PT polynomial, etc)

Applications of classical graph polynomials

As we said before, there are plenty of applications of the classical graph polynomials in

- Graph theory proper and **knot theory**;
- Chemistry and biology;
- Statistical mechanics (Potts and Ising models)
- **Social networks** and **finance mathematics**;
- Quantum physics and quantum computing

And what about the many other graph polynomials

we just learned to construct?

How are the **classical** graph polynomials compared?

There are various ways of comparing graph polynomials:

- By **distinctive power**:

  $P(G;\bar{X}) <_{d.p.} Q(G;\bar{X})$ if for any two graphs G_1, G_2 with $Q(G_1;\bar{X}) = Q(G_2;\bar{X})$ we also have $P(G_1;\bar{X}) = P(G_2;\bar{X})$

- By **coefficient computation**:

  $P(G;\bar{X}) <_{coeff} Q(G;\bar{X})$ if there is a function F which computes for every G the coefficients of $P(G;\bar{X})$ from the coefcients of $Q(G;\bar{X})$.

- By **substitution instance**.

  $P(G;\bar{X}) <_{subst} Q(G;\bar{X})$ if there is a substitution σ of the variables such that for every G $P(G;\bar{X}) = Q(G;\sigma(\bar{X}))$.

Proposition: $P(G;\bar{X}) <_{d.p.} Q(G;\bar{X})$ iff $P(G;\bar{X}) <_{coeff} Q(G;\bar{X})$

If $P(G;\bar{X}) <_{coeff} Q(G;\bar{X})$ then $P(G;\bar{X}) <_{subst} Q(G;\bar{X})$, but not conversely.

How are the **classical** graph polynomials **uniformly** defined?

Most graph polynomials studied in the literature have **several equivalent definitions**:

(i) By **counting generalized colorings**;

(ii) By **generating functions**;

(iii) By **subset expansion** formulas;

(iv) By **recurrence relations**;

(v) By **counting (weighted) homomorphisms**.

We shall see that, by imposing SOL-definability,
(i)-(iii) give the same class of graph polynomials,
whereas (iv) and (v) are special cases thereof.

Complexity of the **classical** graph polynomials?

There are various problems with measuring the complexity of a multi-variate graph polynomial $P(G; \bar{X})$:

Turing Complexity:

Evaluation: Fix $x_0 \in \mathbb{Q}^m$. Measure the complexity of computing $P(G; x_0)$ as a function of the size of G in term is Turing complexity.

Computing the coefficients: Measure the complexity of computing the coefficients of $P(G; x_0)$ as a function of the size of G.

It is usually in EXPTIME, often $\sharp P$-complete, but sometimes in P-Time.

The Turing model does not fit the algebraic character of the problem.

BSS-Complexity: Think of a (weighted) graph being given by its adjacency matrix M_G. Measure the complexity of computing the coefficients of $P(G; \bar{X})$ from the matrix M_G.

It is usually in EXPTIME, but no convincing complexity classes fit the framework.

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The matching polynomials: Case study

The matching polynomials: Case study

We illustrate these concepts with the **bivariate matching polynomial**, as introduced by O.J. Heilman and E.J. Lieb, 1972.

The acyclic polynomial has important applications in Chemistry and molecular Physics of Ferromagnetisms.

It was first studied in the 1970ies (Heilman and Lieb, Kunz)

- L. Lovász and M.D. Plummer
  Matching Theory
  Annals of Discrete mathematics, vol. 29
  North-Holland 1986

- N. Trinajstić,
  Chemical Graph Theory
  CRC, 1992 (2nd edition)

- P.J. Garratt
  Aromaticity
  John Wiley and Sons, 1971

Two univariate matching polynomials

We denote by $m_k(G)$ the number of k-matchings of a graph G,

$$m(G,X) = \sum_{k}^{\frac{n}{2}} (-1)^k m_k(G) X^{n-2k}$$

is called the **acyclic polynomial of G** or the **matching defect polynomial**.

$$g(G,X) = \sum_{k}^{n} m_k(G) X^k$$

is called the **matching generating polynomial of G**.

- It is easy to verify the identity $m(G,X) = X^n g(G,(-X^{-2}))$
- Note that $g(G;X) <_{coeff} m(G;X)$ and hence $g(G;X) <_{d.p.} m(G;X)$, but **not conversely**.

Bivariate matching polynomial

The two matching polynomials are special cases of
the **bivariate matching polynomial**

$$M(G,X,Y) = \sum_{k}^{\frac{n}{2}} X^{n-2k} Y^{k} m_k(G) = \sum_{A} X^{|V(G)|-2|A|} Y^{|A|}$$

where A ranges over all subsets of $E(G)$ which are matchings.

$M(G,X,Y) = \sum_{k}^{\frac{n}{2}} X^{n-2k} Y^{k} m_k(G)$ can be viewed as a **generating function**.

$M(G,X,Y) = \sum_{A} X^{|V(G)|-2|A|} Y^{|A|}$ can be viewed as a **subset expansion**.

Now we have $m(G;X) = M(G;X,-1)$ and $g(G;X) = M(G;1,X)$.

In other words, both $m(G;X)$ and $g(G;X)$ are **substitution instances** of $M(G;X,Y)$.

Interpretation: $\mid A \mid$ is the size of the matching A, and $\mid V(G) \mid -2 \mid A \mid$ is the number of vertices not incident with any edge in A.

The bivariate matching polynomial as a **generalized chromatic** polynomial

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We want to show that the bivariate matching polynomial can be obtained in our framework. We use

- two sorts of colors $[k_1]$ and $[k_2]$;
- a coloring property consisting of 5-tuples

$$\langle V, [k_1], [k_2]; E, r_1, r_2 \rangle,$$

with two coloring relations $r_1 \subseteq E \times [k_1]$ and $r_2 \subseteq V \times [k_2]$ such that

(i) "$r_1 \subseteq E \times [k_1]$ is a partial function the domain M of which is a matching of G" .

(ii) and "$r_2 \subseteq V \times [k_2]$ is a partial function with domain $V - cov(M)$.

Second Order Logic (SOL)

- Second Order Logic is the natural language to talk about graph properties.

  We shall show this informally and only after that define the syntax and semantic of SOL.

- We shall see we can also use SOL to define graph parameters.

Second Order Logic SOL and some of its fragments.

Atomic formulas for graphs are $E(u, v)$ and $u = v$ for individual variables u, v, and $R(u_1, \ldots, u_m)$ for m-ary relation variables R.

- First Order Logic FOL:

  Closed under boolean operations and quantification over individual variables. No relation variables.

- Second order Logic SOL:

  Closed under boolean operations and quantification over individual and relation variables of arbitrary but fixed arity.

- Monadic Second order Logic MSOL:

  Closed under boolean operations and quantification over individual and unary relation variables.

Concrete vs abstract graphs

Skip: Concrete vs abstract graphs

Concrete graphs (in $\mathbb{R}^3$)

A concrete graph G is given by

- a finite set of points V in $\mathbb{R}^3$, and
- a finite set E of ropes linking two points v_1, v_2.
  The ropes are continuous curves which do not intersect.

Without loss of generality we can take the points also in $\mathbb{R}^m$ for $m \geq 3$.

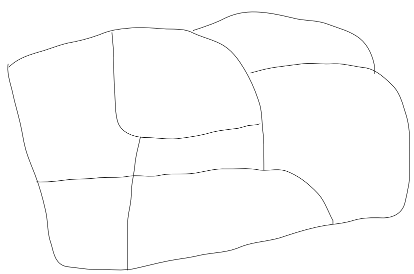
The ropes are called arcs.

Plane graphs

A plane graph G is given by

- a finite set of points V in $\mathbb{R}^2$, and
- finite set E of arcs linking two points v_1, v_2.

The arcs are continuous curves which do not intersect.



All intersection points in the drawing are points of the graph!

Abstract graphs

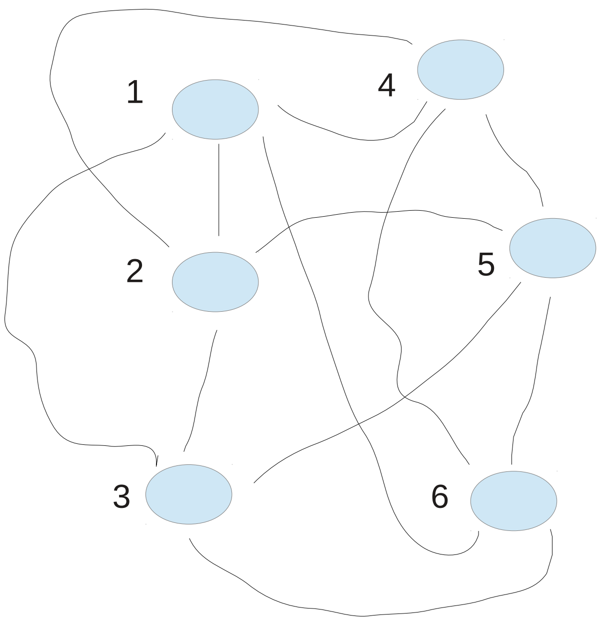
An abstract graph $G = (V(G), E(G))$ is given by

- a finite set of vertices $V = V(G)$, and
- a finite set $E = E(G)$ of edges linking two vertices v_1, v_2.

Here $E \subseteq V^{(2)}$ where $V^{(2)}$ denotes the set of unordered pairs of elements of V.

$$V = \{1, \dots, 6\}$$

$$E = \begin{cases} \{(1,2),(2,3),(3,1)\} \cup \\ \{(4,5),(5,6),(6,4)\} \cup \\ \{(1,6),(6,3),(3,5),(5,2),(2,4),(4,1)\} \end{cases}$$



Graph isomorphism and subgraphs

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a function $f : V_1 \to V_2$ such that

- f is bijective (one-one and onto), and
- $(u, v) \in E_1$ iff $(f(u), f(v)) \in E_2$.

$G_1 = (V_1, E_1)$ is a subgraph of $G_2 = (V_2, E_2)$ if $V_1 \subseteq V_2$ and $E_1 \subseteq V_2$.

$G_1 = (V_1, E_1)$ is an induced subgraph of $G_2 = (V_2, E_2)$ if $V_1 \subseteq V_2$ and for all $(u, v) \in V_1^{(2)} \cap E_2$ we also have $(u, v) \in E_1$.

$G_1 = (V_1, E_1)$ is a spanning subgraph of $G_2 = (V_2, E_2)$ if $E_1 \subseteq E_2$ and for all $(u, v) \in E_1$ both $u, v \in V_1$.

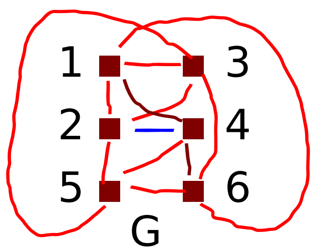
Two isomorphic graphs

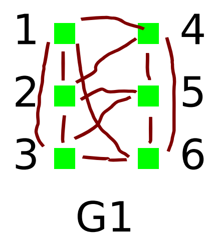
$V_1 = V_2 = \{1, \ldots, 6\}$

$$E_1 = \begin{cases} \{(1,2),(2,3),(3,1),(4,5),(5,6),(6,4)\} \cup \\ \{(1,6),(6,3),(3,5),(5,2),(2,4),(4,1)\} \end{cases}$$

$$E_2 = \begin{cases} \{(1,4),(4,3),(3,1),(5,2),(2,6),(6,5)\} \cup \\ \{(1,6),(6,3),(3,2),(2,4),(4,5),(5,1)\} \end{cases}$$

G_1 and G_2 are isomorphic with
$f(1) = 1, f(2) = 4, f(3) = 3, f(4) = 5, f(5) = 2, f(6) = 6.$

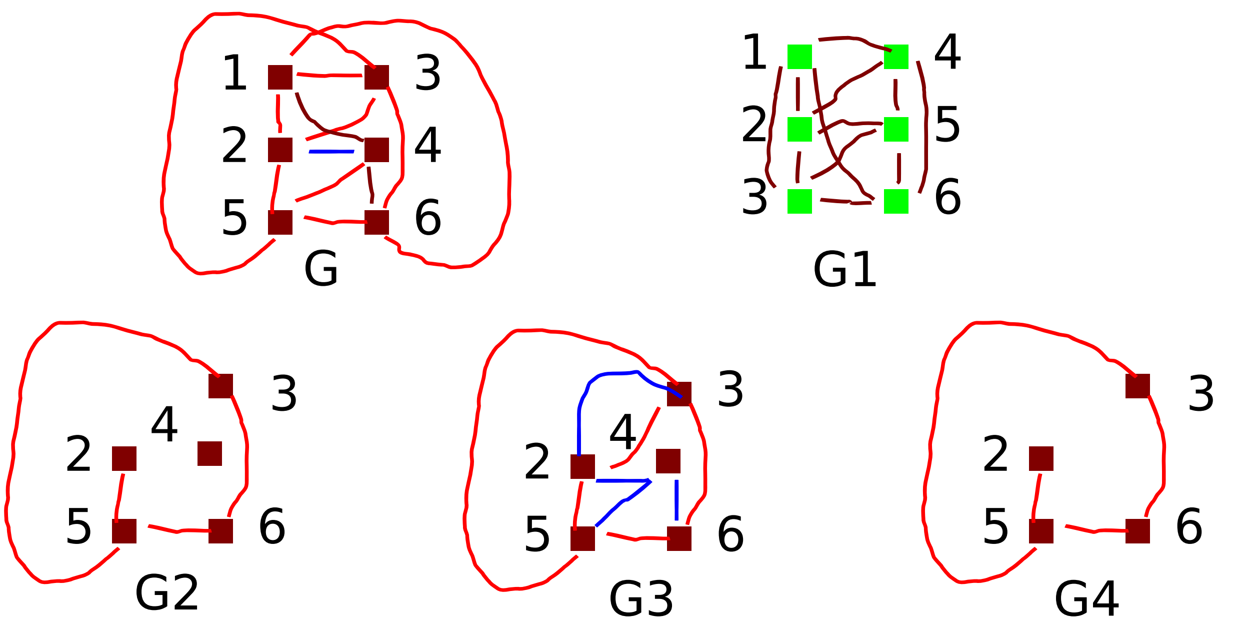




$G1$ is isomorphic to G.

$G2$ is a subgraph of G, but not an induced subgraph.

$G3$ is an induced subgraph and $G4$ is a spanning subgraph of G.



Graph properties

Regularity

Closure Properties

Colorability

Eulerian and Hamiltonian properties

Topological Properties

Graph minors

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Some graph properties: Regularity

A graph G is (give definition in SOL):

- of degree bounded by $d \in \mathbb{N}$.

  Every vertex has at most d neighbors.

- k-regular ($k \in \mathbb{N}$)

  Every vertex has exactly k neighbors.

- regular

  Every vertex has exactly the same number of neighbors.

- Regular and degree bounded by d.

Definable in First Order Logic FOL

- The vertices $v_0, v_1, \dots, v_n$ are all different:

$$\text{Diff}(v_0, v_1, \dots, v_n) : \left(\bigwedge_{i=0, j=1, i<j}^{i,j \leq n} v_i \neq v_j \right)$$

- A vertex v_0 has degree at most d:

$$\text{Deg}_{\leq d}(v_0) : \forall v_1, \dots, v_d, v_{d+1} \left(\bigwedge_{i=0}^{d+1} E(v_0, v_i) \to \bigvee_{i=0, j=0, i \neq j}^{i=d+1, j=d+1} v_i = v_j \right)$$

- A vertex v_0 has degree at least d:

$$\text{Deg}_{\geq d}(v_0) : \exists v_1, \dots, v_d \left(\text{Diff}(v_1, \dots, v_d) \wedge \bigwedge_{i=1}^{d} E(v_0, v_i) \right)$$

Regularity definable in .....

The following graph properties are definable in FOL (use previous slide):

- k-regular;
- regular and of bounded degree d;

The following are not definable in FOL (nor in Monadic Second order Logic MSOL):

- regular;
- each vertex has even degree.

To show non-definability in FOL we need the machinery of

Ehrenfeucht-Fraïssé Games or Connection matrices.

Regularity definable in .....

The following are definable in SOL:

- Two sets $A, B \subseteq V$ have the same size:

  $$\text{EQS}(A, B) : \exists R\,(\text{Funct}(R, A, B) \wedge \text{Inj}(R) \wedge \text{Surj}(R))$$

  where $\text{Funct}(R, A, B), \text{Inj}(R), \text{Surj}(R)$ are FOL-formulas saying that R is a function from A to B which is one-one (injective) and onto (surjective).

- A vertex v has even degree:

  The set of neighbors of v can be partitioned into two sets of equal size

  $$\text{EDeg}(v_0) : \exists A, B\,(\text{Part}(N_v, A, B) \wedge \text{EQS}(A, B))$$

- Two vertices u, v have the same degree:

  The set of neighbors N_u, N_v of u and v have the same size.

  $$\text{SDeg}(u, v) : \text{EQS}(N_u, N_v)$$

Back to properties

Some graph properties: Closure properties of graph classes.

A graph property is called

- hereditary if it is closed under induced subgraphs.
- monotone if it is closed under subgraphs, not necessarily induced.
- monotone decrasing if it is closed under deletion of edges, but not necessarily of vertices.
- monotone increasing if it is closed under addition of edges, but not necessarily of vertices.
- additive if it is closed under disjoint unions.

Note that **monotone** implies **hereditary** and **monotone decreasing**.

Examples for the closure properties

- Regular graphs are only additive.
- Graphs of bounded degree d are monotone and additive.
- Cliques (complete graphs) are hereditary but not monotone.
- Connectivity is only monotone increasing.
- **Exercise:** Check the above closure properties of graph properties for your favorite graph properties.
- **Exercise:** Check the above closure properties of all the graph properties discussed in the sequel of this course.

Back to properties

Forbidden (induced) subgraphs

Let $\mathcal{H} = \{H_i : i \in I\}$ be a family of graphs.

- We denote by Forb$_{sub}(\mathcal{H})$ (Forb$_{ind}(\mathcal{H})$) the class of graphs G which have no (induced) subgraph isomorphic to some graph $H \in \mathcal{H}$.

- Forb$_{sub}(\mathcal{H})$ is monotone and Forb$_{ind}(\mathcal{H})$ is hereditary.

Theorem: (**Exercise**)
Let $\mathcal{P}$ be a monotone (hereditary) graph property. Then there exists a family $\mathcal{H} = \{H_i : i \in I\}$ of finite graphs such that $\mathcal{P} =$ Forb$_{sub}(\mathcal{H})$ (respectively $\mathcal{P} =$ Forb$_{ind}(\mathcal{H})$).

Proposition: Let $\mathcal{H} = \{H_i : i \in I\}$ be a family of graphs with I finite. Then both Forb$_{sub}(\mathcal{H})$ and Forb$_{ind}(\mathcal{H})$ are definable in FOL.

Homework 1

Characterize the following graph properties using
$\mathrm{Forb}_{sub}(\mathcal{H})$ or $\mathrm{Forb}_{ind}(\mathcal{H})$,
and determine their definability in FOL and SOL.

- Forests
- Cliques
- Find other examples! You may consult:

```
@BOOK(bk:BrandstaedtLeSpinrad,
AUTHOR          = {A. Brandst\"adt and V.B. Le and J. Spinrad},
TITLE           = {Graph Classes: A survey},
PUBLISHER       = {{SIAM} },
SERIES  = {{SIAM} Monographs on Discrete Mathematics and Applications},
YEAR            = {1999})
```

Back to properties

Some graph properties: Colorability

Let $\mathcal{P}$ be a graph property. A graph G is (*give definition in* $\mathrm{SOL}, \mathrm{MSOL}$):

- 3-colorable:

  The vertices of G can be partitioned into three disjoint sets $C_i : i = 1, 2, 3$ such that the induced graphs $G[C_i]$ consist only of isolated points.

  This can be expressed in MSOL.

- k-$\mathcal{P}$-colorable($k \in \mathbb{N}$):

  The vertices of G can be partitioned into k disjoint sets $C_i : i = 1, \ldots, k$ such that the induced graphs $G[C_i]$ are in $\mathcal{P}$.

  If $\mathcal{P}$ is definable in SOL (MSOL), this is also definable in SOL (MSOL).

- $\mathcal{P}$-colorable:

  The vertices of G can be partitioned into disjoint sets $C_i : i \in I \subset \mathbb{N}$ such that the induced graphs $G[C_i]$ are in $\mathcal{P}$.

  This is definable in SOL provided $\mathcal{P}$ is. It is not MSOL-definable.

k-colorable graphs

A subset V_1 of a graph $G = (V, E)$ is independent if it induces a graph of isolated points (without neighbors nor loops).

A graph is k-colorable if its vertices can be partitioned into k independent sets.

$$\text{Part}(X_1, X_2, X_3) :$$
$$((X_1 \cup X_2 \cup X_3 = V) \wedge ((X_1 \cap X_2) = (X_2 \cap X_3) = (X_3 \cap X_1) = \emptyset))$$

$$\text{Ind}(X) :$$
$$(\forall v_1 \in X)(\forall v_2 \in X)\neg E(v_1, v_2)$$

With this 3-colorable can be expressed as

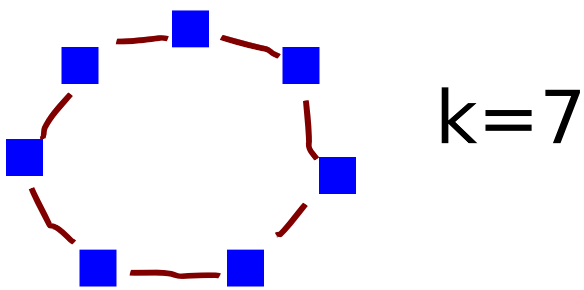
$$\exists C_1 \exists C_2 \exists C_3 \, (Part(C_1, C_2, C_3) \wedge Ind(C_1) \wedge Ind(C_2) \wedge Ind(C_3))$$

We have expressed 3-colorability by a formula in Monadic Second Order Logic.

Question: Can we express this in First Order Logic ?

Some graph properties: Chordality

A graph is a simple cycle of length k of it is of the form:



A graph is a simple cycle iff it is connected and 2-regular.

A graph G is chordal or triangulated if there is no induced subgraph of G isomorphic to a simple cycle of length ≥ 4.

Exercise: Find a MSOL-expression for chordality. Back to properties

Some graph properties: Eulerian and Hamiltonian

A graph G is (*give definition in* SOL):

- Eulerian:

  We can follow each edge exactly once, pass through all the edges, and return to the point of departure.

  Theorem (Euler): A graph is Eulerian iff it is connected and each vertex has even degree.

- Hamiltonian:

  We can follow the edges visiting each vertex exactly once, and return to the point of departure.

Eulerian graphs

A graph $G = (V, E)$ is Eulerian if we can follow each edge exactly once, pass through all the edges, and return to the point of departure.

Equivalently:
Can we order all the edges of E

$$e_1, e_2, e_3, \dots e_m$$

and choose beginning and end of th edge $e_i = (u_i, v_i)$ such that for all i, $v_i = u_{i+1}$ and $v_m = u_1$.

$$\exists R\, (\text{LinOrd}(R, E) \wedge$$
$$(\forall u, v, u', v' \text{First}(R, u, v) \wedge \text{Last}(R, u', v') \rightarrow u = v') \wedge$$
$$(\forall u, v, u', v' \text{Next}(R, u, v, u'v') \rightarrow v = u')\big)$$

whith the obvious meaning of LinOrd(R, E), First(R, u, v) and Last(u, v).

Alternatively, we can use Euler's Theorem.

As we shall see later, it cannot be expressed in MSOL.

Hamiltonian graphs

We note: A graph with n vertices is Hamiltonian if it contains a spanning subgraph which is a cycle of size n.

We define formulas:

Conn(V_1, E_1): (V_1, E_1) is connected.

Cycle(V_1, E_1): (V_1, E_1) is a cycle, i.e., regular of degree 2 and connected.

Ham(V, E) : $\exists V_1 \exists E_1 \, (Cycle(V_1, E_1) \wedge E_1 \subseteq E \wedge V_1 = V)$

A subtle point: Graphs vs hypergraphs, I

- Graphs are structures with universe V of vertices, and a binary edge relation E.
  There can be at most one edge between two vertices.

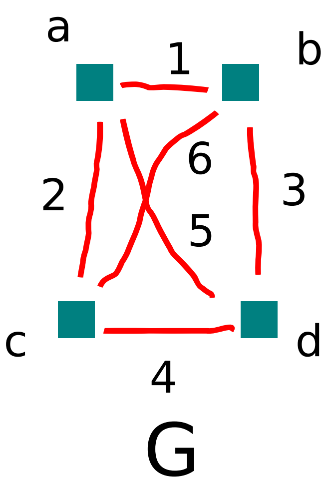
- Hypergraphs have as their universe two disjoint sets V and E and an incidence (hyperedge) relation $R(u, v, e)$.
  There can be many edges between two vertices.

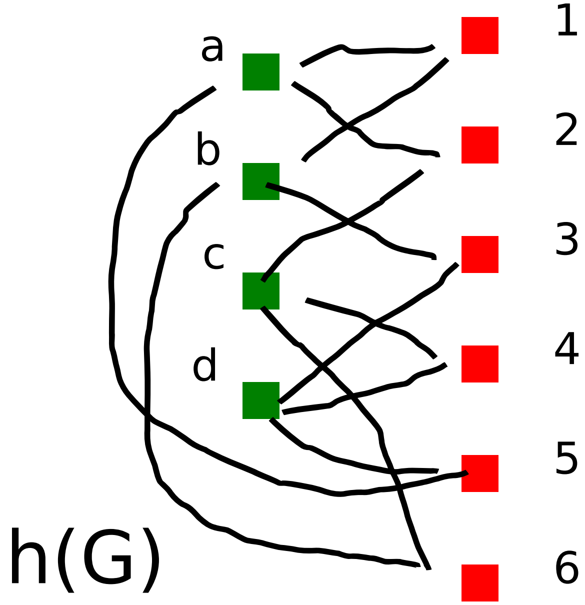
- In both cases the relations are symmetric in the vertices.

- A Graph G can be viewed as hypergraph (h-graph) $h(G)$ where there is at most one edge (up to symmetry) between two vertices.

- There is a one-one correspondence between graph and h-graphs.

G and $h(G)$





A subtle point: Graphs vs hypergraphs, II

- FOL and SOL are equally expressive on graphs and h-graphs.
- MSOL is more expressive on h-graphs than on graphs.

  Hamiltonicity is not definable in MSOL on graphs, but is definable on h-graphs.

We shall discuss this in detail in a later lecture.

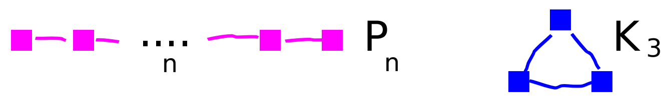
Back to properties

How to prove definablity in SOL, MSOL and FOL?

So far we have looked at properties of
abstract (directed) graphs and hypergraphs.

- Formulate the property using set theoretic language of finite sets over the set of vertices and edges and their incidence relation.
- Try to mimick this formulation in SOL.
- If you succeed, try to do it in MSOL or even FOL.

Test your fluency in SOL! (Homework)



Express the following properties in FOL, if possible.

- A graph G is a cograph if and only if there is no induced subgraph of G isomorphic to a P_4.
- A G is P_4-sparse if no set of 5 vertices induced more than one P_4 in G.
- Triangle-free graphs: There is no induced K_3.
- Existence of prescribed (induced) subgraph H.
- H-free graphs: non-existence of prescribed (induced) subgraph H.
- Let $\mathcal{P}$ be a graph property.
  $\mathcal{P}$-free graphs: non-existence of an induced subgraph $H \in P$.

Topological properties of graphs (from Wikipedia)

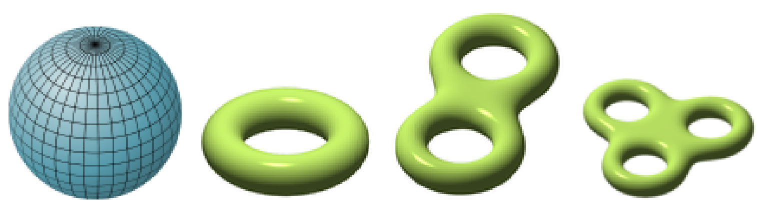
http://en.wikipedia.org/wiki/Genus_(mathematics)

So far our graph properties were formulated in the language of graphs, involving as basic concepts only vertices, edges and their incidence relations.

Topological graph theory studies the embedding of graphs in surfaces, spatial embeddings of graphs, and graphs as topological spaces.

- A graph is planar if it is isomorphic to a plane graph.
- The genus of a graph is the minimal integer n such that the graph can be drawn without crossing itself on a sphere with n handles (i.e. an oriented surface of genus n).

  Thus, a planar graph has genus 0, because it can be drawn on a sphere without self-crossing.



genus: $0, 1, 2, 3$

Planar graphs, I

A graph is planar iff it is isomorphic to a plane graph.

This definition involves the geometry of th Euclidean plane.

How can we express planarity

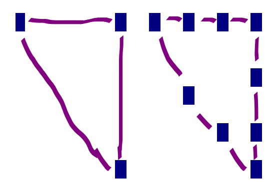
without geometry ?

Kuratowski's Theorem

Kazimierz Kuratowski (1896-1980)

http://en.wikipedia.org/wiki/Kuratowski's_theorem

A subdivision of a graph G is a graph formed by subdividing its edges into paths of one or more edges.



K_3 and a subdivision of K_3

Theorem: A finite graph G is planar if and only if it does not contain a subgraph that is isomorphic to a subdivision of K_5 or $K_{3,3}$.

Planar graphs, II

Theorem: Planarity is definable in MSOL.

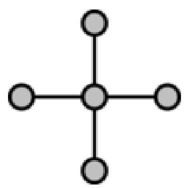
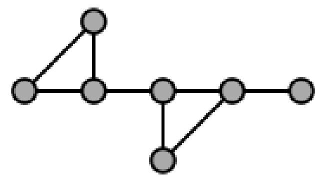
- We use Kuratowski's Theorem.
- For a fixed graph H, G is a subdivision of H, is definable in MSOL.
- For a graph property $\mathcal{P}$ definable in MSOL,
  G has a subgraph $H \in \mathcal{P}$, is definable in MSOL.

Exercise: Prove the last two statements. Back to properties

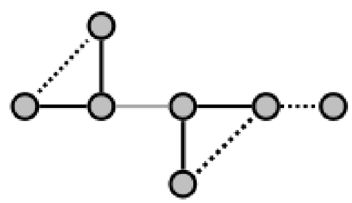
Graph minors, I

http://en.wikipedia.org/wiki/Graph_minor

An undirected graph H is called a minor of the graph G if H can be formed from G by deleting edges and vertices and by contracting edges.

H is a minor of G.



First construct a subgraph of G by deleting the dashed edges (and the resulting isolated vertex), and then contract the thin edge (merging the two vertices it connects).

Graph minors, II

Proposition: For fixed H the statement H is a minor of G is definable in MSOL.

- An edge contraction is an operation which removes an edge from a graph while simultaneously merging the two vertices it used to connect.
- An undirected graph H is a minor of another undirected graph G if a graph isomorphic to H can be obtained from G by contracting some edges, deleting some edges, and deleting some isolated vertices.
- The order in which a sequence of such contractions and deletions is performed on G does not affect the resulting graph H.
- Let $(V)H = \{v_1, \ldots, v_m\}$. We have to find $V_1, \ldots, V_m \subseteq V(G)$ which we all contract to a vertex u_i corresponding to v_i such that V_i connects to V_j iff $(v_i, v_j) \in E(H)$.
- The vertices in $V(G) - \bigcup_i^m V_i$ are discarded.

Minor closed graph classes

- H is a topological minor of G if G has a subgraph which is isomorphic to a subdivision of H.
- A graph property $\mathcal{P}$ is closed under (topological) minors, if whenever $G \in P$ and H is a (topological) minor of G the also $H \in P$.

Examples:

- Trees are not closed under minors, but forests are.
- Graphs of degree at most 2 are minor closed, but graphs of degree at most 3 are not.
- Planar graphs are both closed under minors and topological minors.

Forbidden minors, I

Let $\mathcal{H} = \{H_i : i \in I\}$ be a family of graphs.

- We denote by Forb$_{min}(\mathcal{H})$ (Forb$_{tmin}(\mathcal{H})$) the class of graphs G which have no (topoligical) minors isomorphic to some graph $H \in \mathcal{H}$.

- Forb$_{min}(\mathcal{H})$ is closed under topological minors, is monotone and hence, hereditary.

Theorem: (**Exercise**)
Let $\mathcal{P}$ be a graph property closed under (topological) minors. Then there exists a family $\mathcal{H} = \{H_i : i \in I\}$ of finite graphs such that $\mathcal{P} =$ Forb$_{min}(\mathcal{H})$ (respectively $\mathcal{P} =$ Forb$_{tmin}(\mathcal{H})$).

Proposition: Let $\mathcal{H} = \{H_i : i \in I\}$ be a family of graphs with I finite. Then both Forb$_{min}(\mathcal{H})$ and Forb$_{tmin}(\mathcal{H})$ are definable in MSOL.

The Graph Minor Theorem

(aka Robertson-Seymour Theorem)

Here is one of the deepest theorems in structural graph theory:

Theorem:

Let $\mathcal{P}$ be a graph property closed under minors.

Then $\mathcal{P} = \text{Forb}_{min}(\mathcal{H})$ with $\mathcal{H}$ **finite**.

Corollary: Every graph property $\mathcal{P}$ property closed under minors

is definable in MSOL.

Wagner's Theorem and Hadwiger's Conjecture

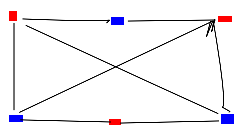
Theorem: A graph G is planar iff K_5 and $K_{3,3}$ are not minors of G.

This gives another proof that planarity is MSOL-definable.

Conjecture: If a graph G is not k-colorable then its has the complete graph K_{k+1} as a minor.

The conjecture was proven for $k \leq 6$.

The converse is not true.
There are bipartite graphs with a K_4 minor.



Back to properties Back to outline 2

Logic and Complexity: Regular languages

Let $L \subseteq \Sigma^\star$ be a language, i.e., a set of words over the alphabet Σ.

We assume you are familiar with automata theory!

Theorem:(Kleene; Büchi, Elgot; Trakhtenbrot)

The following are equivalent:

- L is recognizable by a deterministic finite automaton.
- L is recognizable by a non-deterministic finite automaton.
- L is regular, i.e., describable by a regular expression
- The set of τ_{word}-structures $\mathfrak{A}_w$ with $w \in L$ is definable in MSOL(τ_{word}).

Complexity classes

We need to recall some complexity classes:

L: Deterministic logarithmic space.

NL: Non-deterministic logarithmic space.

P: Deterministic polynomial time.

NP: Non-deterministic polynomial time.

PH: The polynomial hierarchy.

$\sharp$P: Counting predicates in **P** (Valiant's class)

PSpace: Deterministic polynomial space.

Complexity of SOL-properties

Fagin, Christen:
The **NP**-properties of classes of τ-structures are exactly the $\exists SOL$-definable properties.

Meyer,Stockmeyer:
The **PH**-properties (in the *polynomial hierarchy*) of classes of τ-structures are exactly the SOL-definable properties.

Makowsky, Pnueli:
For every level $\mathbf{\Sigma}_n^P$ of $\mathbf{PH}$ there are $MSOL$-definable classes which are complete for it.

Separating Complexity Classes, I

We have

$$\mathbf{L} \subseteq \mathbf{NL} \subseteq \mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PH} \subseteq \sharp\mathbf{P} \subseteq \mathbf{PSpace}$$

- To show that $\mathbf{PH}$ does not collapse to $\mathbf{NP}$ we have to find a τ-sentence ϕSOL(τ) which is not equivalent over finite structures to an existential τ-sentence ψSOL(τ).

- Every sentence $\phi \in SOL(\tau)$ is equivalent (over finite structures) to an existential sentence $\psi \in SOL(\tau)$ iff $\mathbf{NP} = \mathbf{CoNP}$.
  Note we allow arbitrary arities of the quantified relation variables.
  Over infinite structures this is known to be false (Rabin)

- If there is a $\phi \in SOL(\tau)$ which is **not** equivalent to an existential sentence, then $\mathbf{P} \neq \mathbf{NP}$.
  And there should be such a sentence !

- To show that $\mathbf{PSpace}$ is different from $\mathbf{PH}$ it suffices to find a $\mathbf{PSpace}$-complete graph property which is not SOL-definable.

Back to outline 2

HEX and Geography, I

More on HEX.

- The game HEX:
  Given a graph G and two vertices s, t.
  Players I and II color alternately vertices in $V - \{s, t\}$ white and black respectively.
  Player I tries to construct a white path from s to t and Player II tries to prevent this.

  HEX: The class of graphs which allow a Winning Strategy for player I.

- The game GEOGRAPHY:
  Given a **directed** graph G. Players I and II choose alternately new edges starting at the end point of the last chosen edge. The first who cannot find such an edge has lost.

  GEO: The class of graphs which allow a Winning Strategy for I.

HEX and Geography, II

Theorem (Even, Tarjan): HEX is **PSPACE**-complete.

Theorem (Schaefer): GEO is **PSPACE**-complete.

Problem: Are they SOL-definable?

This would imply that **PSPACE= PH**, and the polynomial hierarchy **collapses** to some finite level!

Short versions: Fix $k \in \mathbb{N}$.

SHORT-HEX, SHORT-GEOGRAPHY asks whether Player I can win in k moves.

S-HEX and S-GEO are the class of (orderd) graphs where player I has a winning strategy.

S-HEX and S-GEO are FOL-definable for fixed k.
(and therefore solvable in $\mathbf{P}$).

Shannon switching

The game Shannon switching:
Given a graph G and two vertices s, t.
Players I and II color alternately edges in E white and black respectively.
Player II tries to construct a white path from s to t and Player I tries to prevent this.

ShaSwi: The class of graphs which allow a Winning Strategy for player II.

Theorem 2 (A. Lehmann, 1964)
ShaSwi is SOL*-definable.*

Proof:
The Shannon Switching Game is winning for Player II if and only if the graph contains two edge-disjoint trees on a same subset of vertices containing the two distinguished vertices. Q.E.D.

A. LEHMAN, A solution of the Shannon Switching Game, J. Sot. Zndust. Appl. Math. 12 (1964), 687-725.

Corollary 3
ShaSwi is in $\mathbf{PH}$*.*

A challenge:

Show that that

- HEX
- GEOGRAPHY
- ShaSwi

are **NOT** definable in MSOL!

This may just be achievable with the techniques of the next Lecture.

Back to outline 2

The role of order, I

Let $\tau_=$ be the one sorted vocabulary without any relation or constant symbols. We have only equality as atomic formulas.

Let $\tau_<$ be the one sorted vocabulary with one binary relation symbol $R_<$ which will e interpreted as a linear order.

- The class of structures of even cardinality EVEN is not definable in MSOL($\tau_=$).

  We shall prove this later.

- The class of structures of even cardinality EVEN is definable in MSOL($\tau_<$) with order by a formula ϕ_{EVEN}.

The role of order, II: Constructing ϕ_{EVEN}

We use the order to define the binary relation 2NEXT and the unary relation Odd

- For a structure $\mathfrak{A} = \langle A, < \rangle$, let $(a, b) \in$ 2NEXT$^{\mathfrak{A}}$ iff $a < b$ and there is exactly one element strictly between a and b.
- The first element is in Odd$^{\mathfrak{A}}$.
  If $a \in$ Odd$^{\mathfrak{A}}$ and $(a, b) \in$ 2NEXT$^{\mathfrak{A}}$ then $b \in$ Odd$^{\mathfrak{A}}$.
- Let ϕ_{EVEN} be the formula which says that the last element is not in Odd.
- Now the a structure $\langle A, < \rangle$ is in EVEN iff its last element is not in Odd$^{\mathfrak{A}}$.

Q.E.D.

The role of order, III: Order invariance

In the previous example EVEN the MSOL($\tau_<$)-formula ϕ_{EVEN} is order invariant in the following sense:

Let $\mathfrak{A}_1, \mathfrak{A}_2$ be two structures with universe A and different order relations $<_1$ and $<_2$.
Then $\mathfrak{A}_1 \models \phi_{EVEN}$ iff $\mathfrak{A}_2 \models \phi_{EVEN}$.

We generalise this:

Let $\mathfrak{A}_1, \mathfrak{A}_2$ be two $\tau \cup \{R_<\}$-structures with universe A and different order relations $\mathfrak{A}_1(R_<) =<_1$ and $\mathfrak{A}_2(R_<) =<_2$ but for all other symbols in $R \in \tau$ we have $\mathfrak{A}_1(R) = \mathfrak{A}_2(R)$.

A $\tau \cup \{R_<\}$-formula in SOL is order invariant if for all structures $\mathfrak{A}_1, \mathfrak{A}_2$ as above we have

$$\mathfrak{A}_1 \models \phi \text{ iff } \mathfrak{A}_2 \models \phi$$

The fragment HornESOL(τ).

- A quantifier-free τ-formula is a Horn clause if it is a disjunction of atomic or negated atomic formulas where at most one is not negated.

$$\neg\alpha_1 \vee \neg\alpha_2 \vee \ldots \vee \neg\alpha_n \vee \beta$$

  where α_i, β are atomic.

- A quantifier-free τ-formula is a Horn formula if it is a conjunction of Horn clauses.

- A formula $\phi \in$ SOL(τ) is in HornESOL(τ) of it is of the form

$$\exists U_{1,r_1}, U_{2,r_2}, \ldots, U_{k,r_k} \forall v_1, \ldots, v_m H(v_1, \ldots, v_m, U_{1,r_1}, U_{2,r_2}, \ldots, U_{k,r_k})$$

  where H is a Horn formula and v_i are first order variables.

Some classes of graphs order invariantly (o.i.) definable in HornESOL(τ_{graph})

- Graphs of even cardinality, of even degree. order is needed !
- Bipartite graphs $G = (V_1, V_2, E)$ with $|V_1| = |V_2|$.
- Regular graphs, and regular graphs of even degree.
- Connected graphs.
- Eulerian graphs.

To be discussed on the blackboard.

The Immermann-Vardi-Graedel Theorem (IVG)

Let τ be a relational vocabulary with a binary relation
for the ordering of the universe.

Theorem 4 (Immermann, Vardi, Graedel 1980-4)

Let $\mathcal{C}$ be a set of finite τ-structures. The following are equivalent:

- $\mathcal{C} \in \mathbf{P}$;
- *there is a τ-formula $\phi \in \mathrm{HornESOL}(\tau)$ such that $\mathfrak{A} \in \mathcal{C}$ iff $\mathfrak{A} \models \phi$.*

Here the presence of the ordering is crucial:

Without it the class of structures for the empty vocabulary of even cardinality is in $\mathbf{P}$, but not definable in HornESOL.

The Immermann-Vardi-Graedel Theorem (IVG):

Order invariant version

Let τ be a relational vocabulary and
$\tau_1 = \tau \cup \{R_<\}$. with a binary relation for the ordering of the universe.

Theorem 5 (Graedel 1980-4, Dawar, Makowsky)

Let $\mathcal{C}$ be a set of finite τ-structures. The following are equivalent:

- $\mathcal{C} \in \mathbf{P}$;
- *there is an order invariant τ_1-formula $\phi \in \mathrm{HornESOL}(\tau)$ such that for all τ-structures $\mathfrak{A}$ and linear orderings $R^A \subset \mathfrak{A}(V)^2$ $\mathfrak{A} \in \mathcal{C}$ iff $\langle \mathfrak{A}, R^A \rangle \models \phi$.*

Conclusion: The logical equivalent to $\mathbf{P} = \mathbf{NP}$

Let τ be a relational vocabulary which contains
a binary relation for the ordering of the universe.

The following are equivalent:

- $\mathbf{P} = \mathbf{NP}$ in the classical framework.

- Every ESOL(τ)-formula is equivalent over finite ordered τ-structures to some HornESOL(τ)-formula.

- Every o.i. ESOL(τ)-formula is equivalent over finite ordered τ-structures to some o.i. HornESOL(τ)-formula.

Logics capturing complexity classes

Without requiring the presence of order we have:

- A class $\mathcal{C}$ of finite structures is in $\mathbf{NP}$ iff $\mathcal{C}$ is definable in existential SOL.
- A class $\mathcal{C}$ of finite structures is in $\mathbf{PH}$ iff $\mathcal{C}$ is definable in SOL.

By requiring the presence of an order relation we have

- A class $\mathcal{C}$ of finite structures is in $\mathbf{P}$ iff $\mathcal{C}$ is 0.i. definable in existential HornESOL.
- There are similar theorems for $\mathbf{L}, \mathbf{NL}, \mathbf{PSpace}$.

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Definability of numeric graph invariants and graph polynomials

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We denote by $G = (V(G), E(G))$ a graph,
and by $\mathcal{G}$ and $\mathcal{G}_{simple}$ the class of finite (simple) graphs, respectively.

A **numeric graph invariant** or **graph parameter** is a function

$$f : \mathcal{G} \rightarrow \mathbb{R}$$

which is invariant under graph isomorphism.

(i) Cardinalities: $|V(G)|$, $|E(G)|$

(ii) Counting configurations:

$k(G)$ the number of connected components,
$m_k(G)$ the number of k-matchings

(iii) Size of configurations:

$\omega(G)$ the clique number
$\chi(G)$ the chromatic number

(iv) Evaluations of graph polynomials:

$\chi(G, \lambda)$, the chromatic polynomial, at $\lambda = r$ for any $r \in \mathbb{R}$.
$T(G, X, Y)$, the Tutte polynomial, at $X = x$ and $Y = y$ with $(x, y) \in \mathbb{R}^2$.

Definability of numeric graph parameters, I

We first give examples where we use small, i.e., polynomial sized sums and products:

(i) The cardinality of V is FOL-definable by

$$\sum_{v \in V} 1$$

(ii) The number of connected components of a graph G, $k(G)$ is MSOL-definable by

$$\sum_{C \subseteq V : \text{component}(C)} 1$$

where component(C) says that C is a connected component.

(iii) The graph polynomial $X^{k(G)}$ is MSOL-definable by

$$\prod_{c \in V : \text{first}-\text{in}-\text{comp}(c)} X$$

if we have a linear order in the vertices and first $-$ in $-$ comp(c) says that c is a first element in a connected component.

Definability of numeric graph parameters, II

Now we give examples with possibly large, i.e., exponential sized sums:

(iv) The number of cliques in a graph is MSOL-definable by

$$\sum_{C \subseteq V : \mathrm{clique}(C)} 1$$

where clique(C) says that C induces a complete graph.

(v) Similarly "the number of maximal cliques" is MSOL-definable by

$$\sum_{C \subseteq V : \mathrm{maxclique}(C)} 1$$

where maxclique(C) says that C induces a maximal complete graph.

(vi) The clique number of G, $\omega(G)$ is is SOL-definable by

$$\sum_{C \subseteq V : \mathrm{largest-clique}(C)} 1$$

where largest − clique(C) says that C induces a maximal complete graph of largest size.

Definability of numeric graph parameters, III

Let $\mathcal{R}$ be a (polynomial) ring.

A numeric graph parameter $p : Graphs \to \mathcal{R}$ is $\mathcal{L}$-definable if it can be defined inductively:

- Monomials are of the form $\prod_{\bar{v}:\phi(\bar{v})} t$ where t is an element of the ring $\mathcal{R}$ and ϕ is a formula in $\mathcal{L}$ with first order variables $\bar{v}$.
- Polynomails are obtained by closing under small products, small sums, and large sums.

Usually, summation is allowed over second order variables, whereas products are over first order variables.

$\mathcal{L}$ is typically Second Order Logic or a suitable fragment thereof.
We are especially interested in MSOL and CMSOL, Monadic Second Order Logic, possibly augmented with modular counting quantifiers.

If $\mathcal{L}$ is SOL we denote the definable graphparameters by SOLEVAL$_{\mathcal{R}}$, and similarily for MSOL and CMSOL.

Our definition of SOLEVAL is somehow reminiscent to the defintion of Skolem's definition of the Lower Elementary Functions.

Thank you for your attention

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Outline of the course

LECTURE 00: Second Order Logic (SOL) and its fragments (Background, not lectured)
LOGICS (14 slides)

LECTURE 01: Friday, Oct 10, 2014, 14:00-15:40, Prague Lecture 1,
A landscape of graph parameters and graph polynomials. Comparing graph parameters. Towards a general theory.
(90 minutes, 90 slides with skip-options)

LECTURE 02: Thursday, Oct 16, 2014, 12:20-14:00 Prague Lecture 2,
Why is the chromatic polynomial a polynomial? Where to graph polynomial occur naturally? Definability of graph properties and graph polynomials in fragment of Second Order Logic.
(90 minutes, ca. 99 slides with skip options)

LECTURE 03: Thursday, Oct 16, 2014, 14:30-16:00 Prague Lecture 3,
Connection matrices for graph parameters. When do connection matrices of graph parameters have finite rank? Connection matrices for graph parameters definable in fragments of Second Order Logic. The finite rank theorem. Using connections matrices to prove non-definability.
(90 minutes, ca. 55 slides with skip options)

Further links to the literature.

Further links

[arXiv] J.A. Makowsky's Graph Polynomial Go to **Homepage** at http://www.cs.technion.ac.il/~janos/RESEARCH/gp-homepage.html

[KMR 2013] J. A. Makowsky T. Kotek and E. V. Ravve, **A computational framework for the study of partition functions and graph polynomials**. In Proceedings of the 12th Asian Logic Conference ?11, pages 210?230, 2013. download at http://www.cs.technion.ac.il/~janos/RESEARCH/alcpaper.pdf

[GKM 2012] B. Godlin, E. Katz and J. A. Makowsky, **Graph Polynomials: From Recursive Definitions to Subset Expansion Formulas**. J. Log. Comput. 22(2): 237-265 (2012) download at http://www.cs.technion.ac.il/~janos/RESEARCH/GodlinKatzMakowsky.pdf

[M 2008] J.A. Makowsky, **From a Zoo to a Zoology: Towards a general theory of graph polynomials**, Theory of Computing Systems, 2008. download at http://dx.doi.org/10.1007/s00224-007-9022-9

More links

Further links, II

[arXiv] J.A. Makowsky's papers at `http://arxiv.org/find/all/1/au:+Makowsky/0/1/0/all/0/1?per_page=100` on **arXiv**.

[dblp] J.A. Makowsky's papers at `http://www.informatik.uni-trier.de/~ley/pers/hd/m/Makowsky:Johann_A=.html` on **DBLP**.

[google] J.A. Makowsky's papers at `http://scholar.google.co.il/citations?hl=en&user=ooNKL6UAAAAJ&pagesize=100&view_op=list_works` at **scholar.google**.

[Course notes] J.A. Makowsky's Course notes.

[PhD Theses] PhD Theses on graph polynomials (a selection)

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Further links: Course notes

Slides of courses on graph polynomials and related topics:

Technion 2005/6 Lecture notes of **Advanced Topics in Computer Science (238900)**

Technion 2009/10 Lecture notes of **Advanced Topics in Computer Science (236605)**

Vienna 2014 Lecture notes of **EMCL Lecture 2014: Graph polynomials**

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Further links: PhD Theses

PhD Theses on graph polynomials and related topics:

I. Averbuch PhD Thesis (Technion 2011): **Completeness and Universality Properties of Graph Invariants and Graph Polynomials**

T. Kotek PhD Thesis (Technion 2012): **Definability of combinatorial functions**

M. Trinks PhD Thesis (TU Freiberg 2012): **Graph Polynomials and Their Representations**

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