Lecture 3

Hankel Matrices, Connection Matrices and Definability of Graph Invariants

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An extended set of slides can be found at the **FMT-2012** program page:
`http://www.lsv.ens-cachan.fr/Events/fmt2012/program.php` Program,
`http://www.lsv.ens-cachan.fr/Events/fmt2012/SLIDES/janosmakowsky.pdf` Slides

Motivation and History

of

Hankel matrices

Skip motivation and history

CMSOL-definable graph parameters

- I have developed with various co-authors a framework of **definability** of numeric graph parameters.
  B. Courcelle, B. Godlin, T. Kotek, E. Ravve

- In this talk we discuss a method of proving **non-definability** in Monadic Second Order Logic with modular Counting CMSOL of numeric graph parameters which take values in a field.

- The CMSOL-definable **graph parameters** behave similarily like CMSOL-definable graph properties.

  (i) On graphs of bounded width they are in $\mathbf{FPT}$, where the notion of with and the the notion of monadic quantification have to fit correspondingly.

  (ii) All classical graph polynomials (Tutte polynomial, matching polynomial, chromatic polynomial, interlace polynomial) and many more are CMSOL-definable using order on vertices in an **invariant way**

  (iii) On recursively defined graph sequences (like P_n, C_n, L_n, etc) they can be computed via linear recurrcence relations.

Hankel aka connection matrices

Hankel matrices (over a field $\mathcal{F}$)

Let $f : \mathcal{F} \to \mathcal{F}$ be a function over a field $\mathcal{F}$.

A finite or infinite matrix $H(f) = h_{i,j}$ is a Hankel matrix for f if $H_{i,j} = f(i+j)$.

Hankel matrices have many applications in:
numeric analysis, probability theory and combinatorics.

- Padé approximations
- Orthogonal polynomials
- Probability theory (theory of moments)
- Coding theory (BCH codes, Berlekamp-Massey algorithm)
- Combinatorial enumerations
  (Lattice paths, Young tableaux, matching theory)

Hankel matrices over words

Let Σ be a finite alphabet and $\mathcal{F}$ be a field and let $f : \Sigma^\star \to \mathcal{F}$ be a function on words.

A finite or infinite matrix $H(f) = h_{u,v}$ indexed over the words $u, v \in \Sigma^\star$ is a Hankel matrix for f if $h_{u,v} = f(u \circ v)$. Here $\circ$ denotes concatenation.

Hankel matrices over words have applications in

- Formal language theory and stochastic automata,
  J. Carlyle and A. Paz 1971
- Learning theory (exact learning of queries).
  A.Beimel, F. Bergadano, N. Bshouty, E. Kushilevitz, S. Varricchio 1998
  J. Oncina 2008
- Definability of picture languages.
  O. Matz 1998, and D. Giammarresi and A. Restivo 2008

Hankel matrices for graphs

If we want to define Hankel matrices for (labeled) graphs,

what plays the role of concatenation?

- Disjoint union
  Used by Freedman, Lovász and Schrijver, 2007, for
  characterizing multiplicative graph parameters over the real numbers
- k-unions (connections, connection matrices)
  Used by Freedman, Lovász, Schrijver and Szegedy, 2007ff, for
  characterizing various forms and partition functions.
- Joins, cartesian products, generalized sum-like operations
  used by Godlin, Kotek and JAM (2008) to prove non-definability.

Overview

- Logics and **Definability** of numeric graph invariants
- Non-definability via Complexity Theory
- Typical properties of graph parameters
- **Connection matrices** (aka Hankel matrices) and their **rank**, I
- **Connection matrices** (aka Hankel matrices) and their **rank**, II
- **The Finite Rank Theorem FRT**
- **Many Applications** of the FRT for graph properties
- **Applications** of the FRT for graph polynomials
- **Merits** and **Limitations** of the FRT

Logics

In this talk a logic $\mathcal{L}$ is a fragment of Second Order Logic SOL.

Let $\mathcal{L}$ be a subset of SOL. $\mathcal{L}$ is a fragment of SOL if the following conditions hold.

(i) For every finite relational vocabulary τ the set of $\mathcal{L}(\tau)$ formulas contains all the atomic τ-formulas and is closed under boolean operations and renaming of relation and constant symbols.

(ii) $\mathcal{L}$ is equipped with a notion of quantifier rank and we denote by $\mathcal{L}_q(\tau)$ the set of formulas of quantifier rank at most q. The quantifier rank is subadditive under substitution of subformulas,

(iii) The set of formulas of $\mathcal{L}_q(\tau)$ with a fixed set of free variables is, up to logical equivalence, finite.

(iv) Furthermore, if $\phi(x)$ is a formula of $\mathcal{L}_q(\tau)$ with x a free variable of $\mathcal{L}$, then there is a formula ψ logically equivalent to $\exists x \phi(x)$ in $\mathcal{L}_{q'}(\tau)$ with $q' \geq q + 1$.

(v) A fragment of SOL is called **tame** if it is closed under scalar transductions.

Typical fragments

- First Order Logic FOL.
- Monadic Second Order Logic MSOL.
- Logics augmented by modular counting quantifiers: $D_{m,i}x\phi(x)$ which says that the numbers of elements satisfying ϕ equals i modulo m.
- CFOL, CMSOL denote the logics FOL, resp. MSOL, augmented by all the modular counting quantifiers.
- Logics augmented by Lindström quantifiers.
- Logics restricted a fixed finite set of bound or free variables.

Graph properties (boolean graph invariants)

We denote by $G = (V(G), E(G))$ a graph,
and by $\mathcal{G}$ and $\mathcal{G}_{simple}$ the class of finite (simple) graphs, respectively.

A **graph property** or **boolean graph invariant** is a function

$$f : \mathcal{G} \rightarrow \mathbb{Z}_2$$

which is invariant under graph isomorphism.

More traditionally, a graph property $P = P_f$ is a family of graphs closed under isomorphisms given by $P_f = \{G : f(G) = 1\}$.

(i) P is hereditary, if it is closed under induced subgraphs.

(ii) P is monotone, if it is closed under (not necessarily induced) subgraphs.

(iii) P is definable in some logic $\mathcal{L}$ if there is a formula $\phi \in \mathcal{L}$ such that $P = \{G : G \models \phi\}$.

(iv) Regular graphs of fixed degree d are definable in First order Logic FOL.

(v) Connectivity and planarity are definable in Monadic Second Order Logic MSOL.

Numeric graph invariants (graph parameters)

We denote by $G = (V(G), E(G))$ a graph,
and by $\mathcal{G}$ and $\mathcal{G}_{simple}$ the class of finite (simple) graphs, respectively.

A **numeric graph invariant** or **graph parameter** is a function

$$f : \mathcal{G} \to \mathbb{R}$$

which is invariant under graph isomorphism.

(i) Cardinalities: $|V(G)|$, $|E(G)|$

(ii) Counting configurations:

$k(G)$ the number of connected components,
$m_k(G)$ the number of k-matchings

(iii) Size of configurations:

$\omega(G)$ the clique number
$\chi(G)$ the chromatic number

(iv) Evaluations of graph polynomials:

$\chi(G, \lambda)$, the chromatic polynomial, at $\lambda = r$ for any $r \in \mathbb{R}$.
$T(G, X, Y)$, the Tutte polynomial, at $X = x$ and $Y = y$ with $(x, y) \in \mathbb{R}^2$.

Definability of numeric graph parameters, I

We first give examples where we use small, i.e., polynomial sized sums and products:

(i) The cardinality of V is FOL-definable by

$$\sum_{v \in V} 1$$

(ii) The number of connected components of a graph G, $k(G)$ is MSOL-definable by

$$\sum_{C \subseteq V : \text{component}(C)} 1$$

where component(C) says that C is a connected component.

(iii) The graph polynomial $X^{k(G)}$ is MSOL-definable by

$$\prod_{c \in V : \text{first}-\text{in}-\text{comp}(c)} X$$

if we have a linear order in the vertices and $\text{first} - \text{in} - \text{comp}(c)$ says that c is a first element in a connected component.

Definability of numeric graph parameters, II

Now we give examples with possibly large, i.e., exponential sized sums:

(iv) The number of cliques in a graph is MSOL-definable by

$$\sum_{C \subseteq V : \text{clique}(C)} 1$$

where clique(C) says that C induces a complete graph.

(v) Similarly "the number of maximal cliques" is MSOL-definable by

$$\sum_{C \subseteq V : \text{maxclique}(C)} 1$$

where maxclique(C) says that C induces a maximal complete graph.

(vi) The clique number of G, $\omega(G)$ is is SOL-definable by

$$\sum_{C \subseteq V : \text{largest} - \text{clique}(C)} 1$$

where largest $-$ clique(C) says that C induces a maximal complete graph of largest size.

Definability of numeric graph parameters, III

A numeric graph parameter is $\mathcal{L}$-definable if it can be defined by similar expressions using large and small sums and only small products.

Usually, summation is allowed over second order variables, whereas products are over first order variables.

How can we prove definability and non-definability of graph parameters in some logic $\mathcal{L}$? In particular:

- How to prove that $k(G)$ is not CFOL-definable?
- How to prove that $\omega(G)$ is not CMSOL-definable?
- How to prove that the chromatic number $\chi(G)$ or the chromatic polynomial $\chi(G, X)$ is not CMSOL-definable?

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Non-definability via complexity assumptions:

Harmonious colorings

A vertex coloring of a graph G with k colors is harmonious if it is proper and each pair of colors appears at most once along an edge.

The harmonious index of a graph G is the smallest k such that there is a harmonious coloring with k colors.

- J.E. Hopcroft and M.S. Krishnamoorthy studied harmonious colorings in 1983.
- B. Courcelle, JAM and U. Rotics havee shown that graph parameters (polynomials) definable in CMSOL can be computed in polynomial time for graphs of tree-width at most k.
- K. Edwards and C. McDiarmid showed that computing the harmonious index is $\mathbf{NP}$-hard even on trees.
- So assuming $\mathbf{P} \neq \mathbf{NP}$, the harmonious index is not CMSOL-definable, because trees have tree-width 1.

Non-definability via complexity assumptions: Chromaticity

- B. Courcelle, J.A.M. and U. Rotics proved that graph parameters (polynomials) definable in CMSOL in the language of graphs can be computed in polynomial time for graphs of clique-width at most k.

- The Exponential Time Hypothesis ($\mathbf{ETH}$) says that $3-SAT$ cannot be solved in time $2^{o(n)}$. It was first formulated by R. Impagliazzo, R. Paturi and F. Zane in 2001.

- F. Fomin, P. Golovach, D. Lokshtanov and S. Saurabh proved that, assuming that $\mathbf{ETH}$ holds, the chromatic number $\chi(G)$ cannot be computed in polynomial time.

- Therefore, assuming $\mathbf{ETH}$, the chromatic number and the chromatic polynomial are not CMSOL-definable.

There are many other non-definability results which can obtained like this, for example graph paremeters derived from dominating sets or the size of a maximal cut.

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Our goal is to prove non-definability

without complexity theoretic assumptions.

Additive and multiplicative graph parameters

with respect to a binary operation $\Box$

Let $G_1 \Box G_2$ denote the disjoint union of two graphs.

f is **additive** if $f(G_1 \Box G_2) = f(G_1) + f(G_2)$.
f is **multiplicative** if $f(G_1 \Box G_2) = f(G_1) \cdot f(G_2)$.

For $\Box$ the **disjoint union** we have:

(i) $|V(G)|, |E(G)|, k(G)$ are not multiplicative, but additive.

(ii) $k(G)$ and $b(G)$ are additive.
$b(G)$ is the number of 2-connected components of G.

(iii) $\chi(G)$ and $\omega(G)$ are neither additive nor multiplicative.

(iv) The number of perfect matchings $pm(G)$ is multiplicative and so is the generating matching polynomial $\sum_k m_k(G) X^k$.
Note that $m_k(G)$ is not multiplicative.

(v) The graph polynomials $\chi(G, \lambda)$ and $T(G, X, Y)$ are multiplicative.

Maximizing and minimizing graph parameters

with respect to $\Box$

Let $G_1 \Box G_2$ denote the disjoint union of two graphs.

f is **maximizing** if $f(G_1 \Box G_2) = \max\{f(G_1), f(G_2)\}$.

f is **minimizing** if $f(G_1 \Box G_2) = \min\{f(G_1), f(G_2)\}$.

Again for $\Box$ the **disjoint union** we have

(i) The various chromatic numbers $\chi(G)$, $\chi_e(G)$, $\chi_t(G)$ are maximizing.

(ii) The size of the maximal clique $\omega(G)$ and the maximal degree $\Delta(G)$ are maximizing.

(iii) The tree-width $tw(G)$ and the clique-width $cw(G)$ of a graph are maximizing.

(iv) The minimum degree $\delta(G)$, the girth $g(G)$ are minimizing.

The girth is the minimum length of a cycle in G.

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The connection matrix of a graph parameter

with respect to the disjoint union $\sqcup$

Connection matrix $M(f, \sqcup)$.

Let G_i be an enumeration of all finite graphs (up to isomorphism).

The (full) **connection matrix** $M(f, \sqcup) = m_{i,j}(f, \sqcup)$ is defined by

$$m_{i,j}(f, \sqcup) = f(G_i \sqcup G_j)$$

The **rank** of $M(f, \sqcup)$ is denoted by $r(f, \sqcup)$.

We shall often look at various infinite submatrices of the full connection matrix.

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Examples: Check with $|V(G)|$ and $2^{|V(G)|}$.

Computing $r(f, \sqcup)$

Proposition:

(i) If f is multiplicative, $r(f, \sqcup) = 1$.

(ii) If f is additive, $r(f, \sqcup) = 2$.

(iii) If f is maximizing or minimizing, $r(f, \sqcup)$ is infinite.

(iv) For the average degree $d(G)$ of a graph, $r(d, \sqcup)$ is infinite.

Proof: The first three statements are easy.

For $f = d(G)$ we have

$$M(d, \sqcup) = 2\frac{|E_1| + |E_2|}{|V_1| + |V_2|}.$$

This contains, for graphs with a fixed number e of edges, the Cauchy matrix $(\frac{2e}{i+j})$, hence $r(d, \sqcup)$ is infinite. $\square$.

Characterizing multiplicative graph parameters

M. Freedman, L. Lovász and A. Schrijver, 2007

Theorem: ([FLS] Proposition 2.1.)

Assume f, g are graph parameters with values in an ordered field, and $g(G) \neq 0$ for some graph G.

- $f(G)$ is additive iff $g(G) = 2^{f(G)}$ multiplicative.
- g is multiplicative iff $M(g, \sqcup)$ has rank 1 and is positive semi-definite.

Recall: A finite square matrix M over an ordered field is **positive semi-definite** if for all vectors $\bar{x}$ we have $\bar{x} M \bar{x}^{tr} \geq 0$. An infinite matrix is positive semi-definite, if every finite principal submatrix is positive semi-definite.

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General Connection Matrices (aka Hankel Matrices): I

Let $\mathcal{C}$ be a class possibly labeled graphs, hyper-graphs or τ-structures.

Let $\Box$ be a binary operation defined on $\mathcal{C}$.

Let G_i be an enumeration of all (labeled) finite graphs (structures) in $\mathcal{C}$.

Let f be graph parameter.

The (full) connection matrix $M(f, \Box)$ is defined by

$$M(f, \Box)_{i,j} = f(G_i \Box G_j)$$

and is called the **Full Connection Matrix of f for $\Box$ on $\mathcal{C}$**,

or just a **connection matrix**.

We denote by $r(f, \Box)$ the **rank** of $M(f, \Box)$.

We shall often look at infinite submatrices of $M(f, \Box)$.

Computing $r(f, \Box)$

Proposition:

(i) If f is multiplicative, $r(f, \Box) = 1$.

(ii) If f is additive, $r(f, \Box) = 2$.

(iii) If f is maximizing or minimizing, $r(f, \Box)$ is infinite.

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For $f = d(G)$ we have

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This contains, for graphs with a fixed number e of edges, the Cauchy matrix $(\frac{2e}{i+j})$, hence $r(d, \sqcup)$ is infinite. $\Box$.

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$\mathcal{L}$-smooth operations.

Let $\mathcal{L}$ be a logic.

We say that two graphs G, H are $(\mathcal{L},)q$-equivalent, and write $G \sim_{\mathcal{L}}^{q} H$, if G and H satisfy the same $\mathcal{L}$-sentences of quantifier rank q.

We say that $\Box$ is $\mathcal{L}$-smooth, if wwhenever we have

$$G_i \sim_{\mathcal{L}}^{q} H_i, i = 0, 1$$

then

$$G_0 \Box G_1 \sim_{\mathcal{L}}^{q} H_0 \Box H_1$$

This definition can be adapted to k-ary operations for $k \geq 1$.

Proving that an operation $\Box$ is $\mathcal{L}$-smooth may be difficult.

For FOL this can be achieved using

Ehrenfeucht-Fraïssé games also know as pebble games.

Anther way of establishing smoothness is via the Feferman-Vaught theorem.

Examples of $\mathcal{L}$-smooth operations.

(i) Quantifier-free scalar transductions are both FOL and MSOL-smooth.

(ii) Quantifier-free vectorized transductions are FOL but not MSOL-smooth.

(iii) The cartesian product is FOL-smooth but not MSOL-smooth.

This was shown by A. Mostowski in 1952.

(iv) The (rich) disjoint union is both FOL and MSOL-smooth.

The rich disjoint union has two additional unary predicates to distinguish the universes.

For FOL this was shown by E. Beth in 1952.
For MSOL this is due to H. Läuchli, 1966, using Ehrenfeucht-Fraïssé games

(v) Adding modular counting quantifiers preserves smoothness.

For CMSOL and the disjoint union this is due to B. Courcelle, 1990.
NEW: For $CFOL$ and the **product** this is due to T. Kotek and J.A.M., 2012.

The Finite Rank Theorem

THEOREM (Godlin, Kotek, Makowsky 2008):

Let f be a numeric parameter or polynomial for τ-structures definable in $\mathcal{L}$ and taking values in an integral domain $\mathcal{R}$.

Let $\Box$ be an $\mathcal{L}$-smooth operation.

Then the connection matrix $M(f, \Box)$ has finite rank over $\mathcal{R}$.

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The **Proof** uses a Feferman-Vaught-type theorem for graph polynomials, due to B. Courcelle, J.A.M. and U. Rotics, 2000.

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Applications of the Finite Rank Theorem, I

Disjoint unions

The following graph parameters or not CMSOL-definable

because they are maximizing (minimizing) for the disjoint union.

- $\omega(G)$, the clique number and $\alpha(G)$, the independence number of G.
- The chromatic number $\chi(G)$ and the chromatic index $\chi_e(G)$.
- The degrees $\delta(G)$ (minimal), $\Delta(G)$ (maximal)

The same holds for the average degree $d(G)$, but here we use the fact that the Cauchy matrix growing rank.

Applications of the Finite Rank Theorem, II

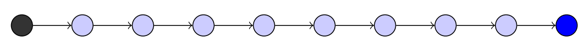
Direct (categorical) products combined with translation schemes

The transduction

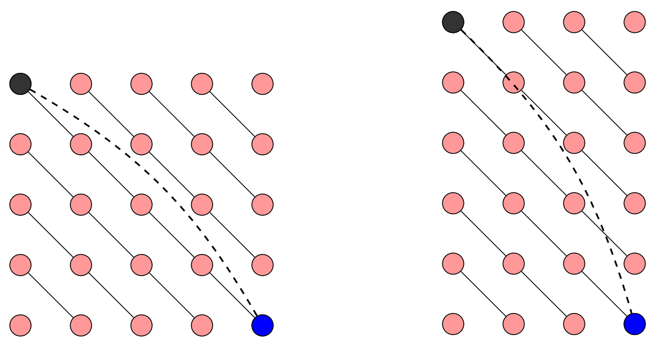
$$\Phi_F((v_1, v_2), (u_1, u_2)) = (E_1(v_1, u_1) \wedge E_2(v_2, u_2)) \vee ((v_1, v_2), (u_1, u_2)) = ((start_1, start_2), (end_1, end_2))$$

transforms the direct product of two directed paths $P^i_{n_i} = (V_1, E_1, start_i, end_i)$ of length n_i with the two constants $start_i$ and end_i, $i = 1, 2$ into an undirected graph with atmost one cycle.

The input graphs look like this:



The result of the transduction is:



$n_1 = n_2$ $n_1 \neq n_2$

THEOREM: Graphs without cycles of odd (even) length

are not CFOL-definable even in the presence of a linear order.

Corollary: Not definable in CFOL with order are

(i) Forests, bipartite graphs, chordal graphs, perfect graphs

(ii) interval graphs (cycles are not interval graphs)

(iii) Block graphs (every biconnected component is a clique)

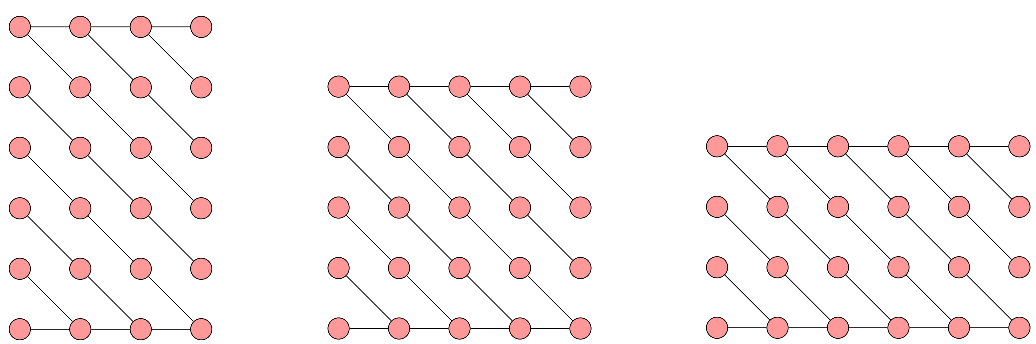
(iv) Parity graphs (any two induced paths joining the same pair of vertices have the same parity)

THEOREM: Trees or connected graphs are not CFOL-definable even in the presence of linear order.

The transduction

$$\begin{aligned}\Phi_T((v_1,v_2),(u_1,u_2)) \quad = \quad & (E_1(v_1,u_1) \wedge E_2(v_2,u_2)) \vee \\ & (v_1 = u_1 = start_1 \wedge E(v_2,u_2)) \vee \\ & (v_1 = u_1 = end_1 \wedge E(v_2,u_2)),\end{aligned}$$

combined with Φ_{sym} transforms the cartesian product of two directed paths into the structures below:



$n_1 > n_2$ $n_1 = n_2$ $n_1 < n_2$

Tree: $n_1 = n_2$. Connected: $n_1 \geq n_2$

k-graphs and k-sums

A k-graph is a graph $G = (V(G), E(G))$

with k distinct vertices labeled with $0, 1, \ldots, k-1$.

Given two k-graphs G_1, G_2 we define the k-sum

$$G_1 \sqcup_k G_2$$

as the disjoint union of G_1 and G_2 where we

identify correspondingly labeled vertices.

Theorem: The k-sum is smooth for $\mathrm{FOL}, \mathrm{CFOL}, \mathrm{MSOL}$ and CMSOL.

THEOREM: Planar graphs are not CFOL-definable even on ordered connected graphs

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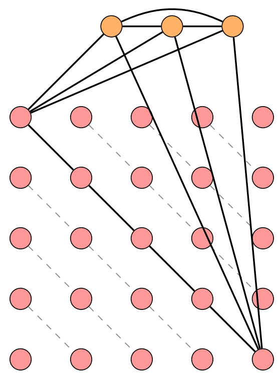
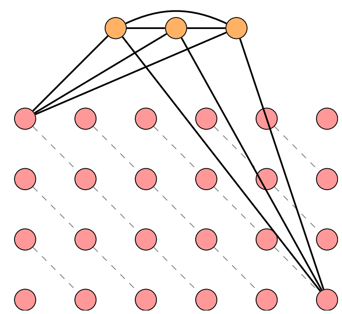
For our next connection matrix we use the 2-sum of the following two 2-graphs:

- the 2-graph (G, a, b) obtained from from K_5 by choosing two vertices a and b and removing the edge between them
- the cartesian product of the two graphs $P^1_{n_1}$ and $P^2_{n_2}$:



$n_1 = n_2$



$n_1 \neq n_2$

The result of this construction has a clique of size 5 as a minor iff $n_1 = n_2$.

It can never have a $K_{3,3}$ as a minor.

A modification ....

If we modify the above construction by taking K_3 instead of K_5 and making $(start_1, start_2)$ and (end_1, end_2) adjacent, we get

Proposition: The following classes of graphs are not CFOL-definable even on ordered connected graphs.

(i) Cactus graphs, i.e. graphs in which any two cycles have at most one vertex in common.

(ii) Pseudo-forests, i.e. graphs in which every connected components has at most one cycle.

Non-definability in CMSOL for graphs $G = (V, E)$

Using the join operation

The join operation of graphs $G = (V, E)$, where E is the edge relation, is defined by

$$(V_1, E_1) \bowtie (V_2, E_2) = (V_1 \sqcup V_2, E_1 \sqcup E_2 \cup \{(v_1, v_2) : v_1 \in V_1, v_2 \in V_2\}$$

This is a quantifier free transduction of the disjoint union, hence smooth for CMSOL.

Consider the connection matrix where the rows and columns are labeled by the graphs on n vertices but without edges E_n.

The graph $E_i \bowtie E_j = K_{i,j}$ is

- hamiltonian iff $i = j$;
- has a perfect matching iff $i = j$;
- is a cage graph (a regular graph with as few vertices as possible for its girth) iff $i = j$;
- is a well-covered graph (every minimal vertex cover has the same size as any other minimal vertex cover) iff $i = j$.

All of these connection matrices have infinite rank.

Proposition: None of the properties above are CMSOL-definable as graphs even in the presence of an order.

CMSOL for hyper-graphs $G = (V, E; R)$

A hyper-graphs $G = (V, E; R)$ has vertices V and edges E

and an incidence relation R between the two.

- CMSOL for hyper-graphs $G = (V, E; R)$ allows quantification over edge sets.
- For the language of hypergraphs the join operation is

  neither MSOL- nor CMSOL-smooth,

  since it increases the number of edges.
- Note also that hamiltonicity and having a perfect matching are both definable in CMSOL in the language of hypergraphs.

In the many papers of B. Courcelle, MSOL on graphs is called $MSOL_1$

and for hyper-graphs it is called $MSOL_2$.

Non-definability in CMSOL for hyper-graphs $G = (V, E; R)$

Using the disjoint union

Using the connection submatrices of the disjoint union we still get:

- Regular: $K_i \sqcup K_j$ is regular iff $i = j$;
- A generalization of regular graphs are *bidegree* graphs, i.e., graphs where every vertex has one of two possible degrees. $K_i \sqcup (K_j \sqcup K_1)$ is a bidegree graph iff $i = j$.
- The average degree of $K_i \sqcup E_j$ is at most $\frac{|V|}{2}$ iff $i = j$;
- A digraph is *aperiodic* if the common denominator of the lengths of all cycles in the graph is 1. We denote by C_i^d the directed cycle with i vertices. For prime numbers p, q the digraphs $C_p \sqcup C_q$ is aperiodic iff $p \neq q$.
- A graph is *asymmetric (or rigid)* if it has no non-trivial automorphisms. It was shown by P. Erdös and A. Rényi (1963) that almost all finite graphs are asymmetric. So there is an infinite set $I \subseteq \mathbb{N}$ such that for $i \in I$ there is an asymmetric graph R_i of cardinality i. $R_i \sqcup R_j$ is asymmetric iff $i \neq j$.

Proposition: None of the properties above are CMSOL-definable as hyper-graphs even in the presence of an order.

Bach to outline of Lecture 3

The harmonious chromatic polynomial

Recall: A vertex coloring of a graph G with k colors is harmonious if it is proper and each pair of colors appears at most once along an edge.

$\chi_{harmonious}(G;k)$ counts the number of harmonious colorings of G with at most k colors.

The harmonious index $\chi_{harmonious}(G)$ of a graph G is the smallest k such that there is a harmonious coloring with k colors.

Let iP_2 be the graph which consists of i disjoint edges (in the language of hyper-graphs.

Proposition:

(i) $\chi_{harmonious}(iP_2 \sqcup jP_2, k) = 0$ iff $i + j > \binom{k}{2}$.

(ii) $\chi_{harmonious}(iP_2 \sqcup jP_2) = \min_k\{i + j \leq \binom{k}{2}$.

(iii) $\chi_{harmonious}(G;k)$ is **not** CMSOL-**definable** in the language of hyper-graphs.

(iv) $\chi_{harmonious}(G)$ is **not** CMSOL-**definable** in the language of hyper-graphs.

Three graph polynomials, I

Rainbow polynomial $\chi_{rainbow}(G,k)$ is the number of path-rainbow connected k-colorings, which are functions $c : E(G) \to [k]$ such that between any two vertices $u, v \in V(G)$ there exists a path where all the edges have different colors.

MCC-polynomial For every fixed $t \in \mathbb{N}$, $\chi_{mcc(t)}(G,k)$ is the number of vertex k-colorings $f : V(G) \to [k]$ for which every color induces a subgraph with a **c**onnected **c**omponent of **m**aximal size t.

Convex coloring polynomial $\chi_{convex}(G,k)$ is the number of convex colorings, i.e., vertex k-colorings $f : V(G) \to [k]$ such that every color induces a connected subgraph of G.

Makowsky and B. Zilber (2005) showed that $\chi_{rainbow}(G,k)$, $\chi_{mcc(t)}(G,k)$, and $\chi_{convex}(G,k)$ are graph polynomials with k as the variable.

Path-rainbow connected colorings were introduced by G. Chartrand et al. in 2008.
Their complexity was studied in S. Chakraborty et. al in 2008. $mcc(t)$-colorings were first studied by N. Alon et al. in 2003.
Note $\chi_{mcc(1)}(G,k)$ is the chromatic polynomial.

Convex colorings were studied by S. Moran in 2007.

Three graph polynomials, II

Proposition: The following connection matrices have infinite rank:

(i) $M(\sqcup_1, \chi_{rainbow}(G,k))$;

(ii) $M(\sqcup_1, \chi_{convex}(G,k))$;

(iii) For every $t > 0$ the matrix $M(\bowtie, \chi_{mcc(t)}(G,k))$;

Proof:

$\chi_{rainbow}(G,k)$**:** We use that the 1-sum of paths with one end labeled is again a path with $P_i \sqcup_1 P_j = P_{i+j-1}$ and that $\chi_{rainbow}(P_r,k) = 0$ iff $r > k+3$.

$\chi_{convex}(G,k)$**:** We use edgeless graphs and disjoint union $E_i \sqcup E_j = E_{i+j}$ and that $\chi_{convex}(E_r,k) = 0$ iff $r > k$.

$\chi_{mcc(t)}(G,k)$**:** We use the join and cliques, $K_i \bowtie K_j = K_{i+j}$ and that $\chi_{mcc(t)}(K_r,k) = 0$ iff $r > kt$.

Three graph polynomials, III

Corollary:

(i) $\chi_{rainbow}(G, k)$ and $\chi_{convex}(G, k)$ are not an CMSOL-definable in the language of graphs and hypergraphs.

(ii) $\chi_{mcc(t)}(G, k)$ (for any fixed $t > 0$) is not CMSOL-definable in the language of graphs.

(iii) In particular the chromatic polynomial is not CMSOL-definable in the language of graphs.

Note: It is however CMSOL-definable in the language of ordered hypergraphs.

Proof:
(i) The 1-sum and the disjoint union are CMSOL-sum-like and CMSOL-smooth for hypergraphs.
(ii) The join is only CMSOL-sum-like and CMSOL-smooth for graphs.

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Proving non-definability with connection matrices: Merits

The advantages of the Finite Rank Theorem for tame $\mathcal{L}$ in proving that a property is not definable in $\mathcal{L}$ are the following:

(i) It suffices to prove that certain binary operations on graphs (τ-structures) are $\mathcal{L}$-smooth operation.

(ii) Once the $\mathcal{L}$-smoothness of a binary operation has been established, proofs of non-definability become surprisingly simple and transparent.

One of the most striking examples is the fact that asymmetric (rigid) graphs are not definable in CMSOL.

(iii) Many properties can be proven to be non-definable using the same or similar submatrices, i.e., matrices with the same row and column indices. This was well illustrated in the shown examples.

Proving non-definability with connection matrices: Limitations

The classical method of proving non-definability in FOL using pebble games is complete in the sense that a property is FOL$(\tau)_q$-definable iff the class of its models is closed under game equivalence of length q.

Using pebble games one proves easily that the class of structures without any relations of even cardinality, EVEN, is not FOL-definable.

However, one cannot prove that EVEN is not FOL-definable using infinite rank connection matrices, in the following sense:

Proposition: Let Φ a quantifierfree transduction between τ-structures and let $\Box_\Phi$ be the binary operation on τ-structures:

$$\Box_\Phi(\mathfrak{A}, \mathfrak{B}) = \Phi^\star(\mathfrak{A} \sqcup_{rich} \mathfrak{B})$$

Then the connection matrix $M(\Box_\Phi, \text{EVEN})$ satisfies:

(i) There is a finite partition $\{U_1, \ldots, U_k\}$ of the (finite) τ-structures such that the submatrices obtained from restricting $M(\Box, \psi)$ to $M(\text{EVEN}, \Box_\Phi)^{[U_i, U_j]}$ have constant entries.

(ii) In particular, the infinite matrix $M(\text{EVEN}, \Box_\Phi)$ has finite rank over any field $\mathcal{F}$.

(iii) $M(\text{EVEN}, \Box_\Phi)$ has an infinite submatrix of rank at most 1.

Note that EVEN is trivially definable in CFOL.

References, I

[KMZ-2011] T. Kotek and J.A. Makowsky and B. Zilber,
On Counting Generalized Colorings,
in: Model Theoretic Methods in Finite Combinatorics,
M. Grohe and J.A. Makowsky eds,
Contemporary Mathematics, vol 558 (2011) pp. 207-242
American Mathematical Society

[Kotek-Thesis] T. Kotek,
Definability of combinatorial functions,
Ph.D. Thesis, Submitted: March 2012,
Technion - Israel Institute of Technology, Haifa, Israel

[KM-2012] B. Godlin, T. Kotek and J.A. Makowsky
Evaluations of Graph Polynomials.
WG 2008: 183-194

[KM-2014] T. Kotek and J.A. Makowsky
Connection Matrices and the Definability of Graph Parameters
arXiv:1308.3654 (to appear in *Logical Methods in Computer Science*

Thank you for your attention

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Back to outline 1 of Prague Lectures

Outline of the course

LECTURE 00: Second Order Logic (SOL) and its fragments (Background, not lectured)
LOGICS (14 slides)

LECTURE 01: Friday, Oct 10, 2014, 14:00-15:40, Prague Lecture 1,
A landscape of graph parameters and graph polynomials. Comparing graph parameters. Towards a general theory.
(90 minutes, 90 slides with skip-options)

LECTURE 02: Thursday, Oct 16, 2014, 12:20-14:00 Prague Lecture 2,
Why is the chromatic polynomial a polynomial? Where to graph polynomial occur naturally? Definability of graph properties and graph polynomials in fragment of Second Order Logic.
(90 minutes, ca. 99 slides with skip options)

LECTURE 03: Thursday, Oct 16, 2014, 14:30-16:00 Prague Lecture 3,
Connection matrices for graph parameters. When do connection matrices of graph parameters have finite rank? Connection matrices for graph parameters definable in fragments of Second Order Logic. The finite rank theorem. Using connections matrices to prove non-definability.
(90 minutes, ca. 55 slides with skip options)

Further links to the literature.

Further links

[arXiv] J.A. Makowsky's Graph Polynomial Go to **Homepage** at http://www.cs.technion.ac.il/~janos/RESEARCH/gp-homepage.html

[KMR 2013] J. A. Makowsky T. Kotek and E. V. Ravve, **A computational framework for the study of partition functions and graph polynomials**. In Proceedings of the 12th Asian Logic Conference ?11, pages 210?230, 2013. download at http://www.cs.technion.ac.il/~janos/RESEARCH/alcpaper.pdf

[GKM 2012] B. Godlin, E. Katz and J. A. Makowsky, **Graph Polynomials: From Recursive Definitions to Subset Expansion Formulas**. J. Log. Comput. 22(2): 237-265 (2012) download at http://www.cs.technion.ac.il/~janos/RESEARCH/GodlinKatzMakowsky.pdf

[M 2008] J.A. Makowsky, **From a Zoo to a Zoology: Towards a general theory of graph polynomials**, Theory of Computing Systems, 2008. download at http://dx.doi.org/10.1007/s00224-007-9022-9

More links

Further links, II

[arXiv] J.A. Makowsky's papers at `http://arxiv.org/find/all/1/au:+Makowsky/0/1/0/all/0/1?per_page=100` on **arXiv**.

[dblp] J.A. Makowsky's papers at `http://www.informatik.uni-trier.de/~ley/pers/hd/m/Makowsky:Johann_A=.html` on **DBLP**.

[google] J.A. Makowsky's papers at `http://scholar.google.co.il/citations?hl=en&user=ooNKL6UAAAAJ&pagesize=100&view_op=list_works` at **scholar.google**.

[Course notes] J.A. Makowsky's Course notes.

[PhD Theses] PhD Theses on graph polynomials (a selection)

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Further links: Course notes

Slides of courses on graph polynomials and related topics:

Technion 2005/6 Lecture notes of **Advanced Topics in Computer Science (238900)**

Technion 2009/10 Lecture notes of **Advanced Topics in Computer Science (236605)**

Vienna 2014 Lecture notes of **EMCL Lecture 2014: Graph polynomials**

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Further links: PhD Theses

PhD Theses on graph polynomials and related topics:

I. Averbuch PhD Thesis (Technion 2011): **Completeness and Universality Properties of Graph Invariants and Graph Polynomials**

T. Kotek PhD Thesis (Technion 2012): **Definability of combinatorial functions**

M. Trinks PhD Thesis (TU Freiberg 2012): **Graph Polynomials and Their Representations**

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