

## Logics, a reminder

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We define logics.

- Vocabularies: The [basic relations](#)
- Structures: [Interpretations of vocabularies](#)
- Variables: Individual variables, relation variables, function variables
- Atomic formulas
- Boolean closures
- Quantifications

## Vocabularies

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A **vocabulary** is a (finite) set of **basic symbols**.

We deal with (possibly **many-sorted**) relational vocabularies.  
The basic symbols are **sorts symbols** and **relation symbols**.

**Sort symbols:**  $U_\alpha : \alpha \in \mathbb{N}$

**Relation symbols:**  $R_{i,\alpha} : i \in \text{Ar}, \alpha \in \mathbb{N}$  where  $\text{Ar}$  is a set of **arities**, i.e. of finite sequences of sort symbols.

**Constant symbols:**  $c_{\alpha,\beta}$  for  $\alpha, \beta \in \mathbb{N}$ , where  $\alpha$  indicates the sort number.

In the case of one-sorted vocabularies, the arity is just of the form  $\underbrace{\langle U, U, \dots, U \rangle}_n$  which will be denoted by  $n$ .

Vocabularies are denoted by greek letters  $\tau, \sigma, \tau_i, \sigma_i$  with  $i \in \mathbb{N}$ .

## $\tau$ -structures, I

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$\tau$ -structures are [interpretations of vocabularies](#).

More precisely, a  $\tau$ -structure is a function assigning subsets of cartesian products of a fixed set  $A$  to each symbol.

$$\mathfrak{A} : \tau \rightarrow A \cup \bigcup_{n=1}^{\infty} \wp(A^n)$$

with the following restrictions:

- $\mathfrak{A}(U_\alpha) = A_\alpha \subseteq A$
- $\mathfrak{A}(U_\alpha) \cap \mathfrak{A}(U_\beta) = \emptyset$  for  $\alpha \neq \beta$
- If  $i = \langle U_{\alpha_1}, \dots, U_{\alpha_k} \rangle$  is the arity of  $R_{i,\alpha}$  then  $\mathfrak{A}(R_{i,\alpha}) \subseteq A_{\alpha_1} \times \dots \times A_{\alpha_k}$
- $\mathfrak{A}(c_{\alpha,\beta}) \in A_\alpha$ .

## $\tau$ -structures, II: Graphs and hypergraphs

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**Graphs and digraphs:**  $\tau_{graph} = \{U_1, R_{2,1}\}$ .

The elements of the set  $\mathfrak{A}(U_1) = V$  are called **vertices**. The subset  $\mathfrak{A}(R_{2,1}) = E \subseteq V^2$  is called the **(directed) edge relation**.

If  $E$  is symmetric, the  $\tau$ -structures is an **undirected graph**, otherwise it is a **directed graph (aka digraph)**.

If  $(u, u) \in E$  the vertex  $u$  has a **loop**.

**Hypergraphs:**  $\tau_{hgraph} = \{U_1, U_2, R_{\langle 1,2 \rangle, 1}\}$

The elements of the set  $\mathfrak{A}(U_1) = V$  are called **vertices**.

The elements of the set  $\mathfrak{A}(U_2) = E$  are called **edges**.

The subset  $\mathfrak{A}(R_{\langle 1,2 \rangle, 1}) \subseteq V \times E$  is called the **undirected incidence relation**.

**Directed hypergraphs:**  $\tau_{hgraph} = \{U_1, U_2, R_{\langle 1,2,1 \rangle, 1}\}$

The elements of the set  $\mathfrak{A}(U_1) = V$  are called **vertices**.

The elements of the set  $\mathfrak{A}(U_2) = E$  are called **edges**.

The subset  $\mathfrak{A}(R_{\langle 1,2,1 \rangle, 1}) \subseteq V \times E \times V$  is called the **directed incidence relation**.

## $\tau$ -structures, III: Labeled graphs and words

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**Vertex labeled Graphs:** Graphs with  $\ell$ -many **vertex labels**,  $\ell \in \mathbb{N}$ :

$$\tau_{lgraph} = \{U_1, R_{2,1}, P_1, \dots, P_\ell\},$$

like graphs but with unary predicates  $P_i$  for **vertex labels**.

**Edge labeled Graphs:** Graphs with  $\ell$ -many **edge labels**,  $\ell \in \mathbb{N}$ :

$$\tau_{lgraph} = \{U_1, R_{2,i}\} \text{ with } i = 1, \dots, \ell,$$

like graphs but with  $\ell$ -many edge relations for **edge labels**.

**Words in  $\Sigma^*$ :** Let  $\Sigma$  be a finite alphabet (set).

$$\tau_{word} = \{U_1, R_{2,1}, R_{1,a}\}, a \in \Sigma, \text{ where}$$

$\mathfrak{A}(R_{2,1})$  is a **linear order**, and

$$\mathfrak{A}(R_{1,a}) \cap \mathfrak{A}(R_{1,b}) = \emptyset \text{ for } a, b \in \Sigma, a \neq b, \text{ and } \bigcap_{a \in \Sigma} \mathfrak{A}(R_{1,a}) = \mathfrak{A}(U_1).$$

$\tau_{word}$ -structures satisfying these conditions are **words in  $\Sigma^*$** .

## Empty structures

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In **logic** and **universal algebra** a  $\tau$ -structure  $\mathfrak{A}$  is **non-empty**, i.e., for at least on sort symbol  $U_\alpha \in \tau$  the set  $\mathfrak{A}(U_\alpha) \neq \emptyset$ .

**We allow empty structures!**

The reason for not allowing empty structures is the axiomatization of First Order Logic FOL. The axiom

$$\forall x P(x) \Rightarrow \exists x P(x)$$

only holds in **non-empty blue-sorted**  $\tau$ -structures.

## Making structures one-sorted

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We can always make  $\tau$ -structures into **one-sorted**  $\tau'$ -structures:

- We replace the sorts  $U_\alpha \in \tau$  by one sort  $V \in \tau'$ .
- We add for each sort  $U_\alpha \in \tau$  a unary relation symbol  $P_\alpha \in \tau'$ .
- We replace each  $R_{(\alpha_1, \dots, \alpha_m), i} \in \tau$  by  $R_{m, i} \in \tau'$ . Constant symbols remain the same.

We then make a  $\tau$ -structure  $\mathfrak{A}$  into a  $\tau'$ -structure  $\mathfrak{A}'$  by setting

- $\mathfrak{A}'(V) = \bigcup_{U_\alpha \in \tau} \mathfrak{A}(U_\alpha)$ , and
- $\mathfrak{A}'(P_\alpha) = \mathfrak{A}(U_\alpha)$
- $\mathfrak{A}'(R_{(\alpha_1, \dots, \alpha_m), i}) = \mathfrak{A}'(R_{m, i})$

## Isomorphisms and homomorphisms of $\tau$ -structures

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Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be two  $\tau$ -structures on sets

$A = \bigcup_{\alpha, U_\alpha \in \tau} \mathfrak{A}(U_\alpha)$  and  $B = \bigcup_{\alpha, U_\alpha \in \tau} \mathfrak{B}(U_\alpha)$  respectively.

Let  $f : A \rightarrow B$  a function.  $f$  is a  $\tau$ -homomorphism if

- For all  $U_\alpha \in \tau$  we have:  
 $a \in \mathfrak{A}(U_\alpha)$  iff  $f(a) \in \mathfrak{B}(U_\alpha)$ .
- For all  $R_{(\alpha_1, \dots, \alpha_m), i} \in \tau$  we have:  
 $(a_1, \dots, a_m) \in \mathfrak{A}(R_{(\alpha_1, \dots, \alpha_m), i})$  iff  $(f(a_1), \dots, f(a_m)) \in \mathfrak{B}(R_{(\alpha_1, \dots, \alpha_m), i})$ .
- For all  $c_\alpha \in \tau$  we have:  
 $f(\mathfrak{A}(c_\alpha)) = \mathfrak{B}(c_\alpha)$ .

$f$  is a  $\tau$ -isomorphism if additionally  $f$  is one-one and onto.

$\mathfrak{A}$  and  $\mathfrak{B}$  are  $\tau$ -isomorphic if there is a  $\tau$ -isomorphism  $f : A \rightarrow B$ .

## $\tau$ -substructures

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Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be two  $\tau$ -structures on sets  $A = \bigcup_{\alpha, U_\alpha \in \tau} \mathfrak{A}(U_\alpha)$  and  $B = \bigcup_{\alpha, U_\alpha \in \tau} \mathfrak{B}(U_\alpha)$  respectively.

$\mathfrak{A}$  is isomorphic to a **substructure** of  $\mathfrak{B}$  if there is a function  $f : A \rightarrow B$  such that:

- $f$  is one-one.
- For all  $U_\alpha \in \tau$  we have:  
If  $a \in \mathfrak{A}(U_\alpha)$  then  $f(a) \in \mathfrak{B}(U_\alpha)$ .
- For all  $R_{(\alpha_1, \dots, \alpha_m), i} \in \tau$  we have:  
If  $(a_1, \dots, a_m) \in A^m$  then  
 $(a_1, \dots, a_m) \in \mathfrak{A}(R_{(\alpha_1, \dots, \alpha_m), i})$  iff  $(f(a_1), \dots, f(a_m)) \in \mathfrak{B}(R_{(\alpha_1, \dots, \alpha_m), i})$ .
- For all  $c_\alpha \in \tau$  we have:  
 $f(\mathfrak{A}(c_\alpha)) = \mathfrak{B}(c_\alpha)$ .

If  $f$  is the identity, we say  $\mathfrak{A}$  is a **substructure** of  $\mathfrak{B}$ .

## Subgraphs and induced subgraphs

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In graph theory an undirected graph  $G$  **without multiple edges** is given by two sets  $V(G)$  and  $E(G)$  with  $E(G) \subseteq V(G)^{(2)}$ .

Let  $G, H$  be two graphs.

**Subgraph:**  $H$  is a **subgraph** of  $G$  if  $V(H) \subseteq V(G)$  and  $E(H) \subseteq V(H)^2 \cap E(G)$ .

This corresponds to the notion of **substructure** for graphs viewed as **hypergraphs**. i.e.,  $\tau$ -structures for  $\tau = \tau_{hgraph}$

**Induced subgraph:**  $H$  is an **induced subgraph** of  $G$  if  $V(H) \subseteq V(G)$  and  $E(H) = V(H)^{(2)} \cap E(G)$ .

This corresponds to the notion of **substructure** for graphs viewed as **graphs**, i.e.,  $\tau$ -structures for  $\tau = \tau_{graph}$

**Isomorphisms:**  $H$  and  $G$  are isomorphic as  $\tau_{graph}$ -structures iff they are isomorphic as  $\tau_{hgraph}$ -structures.

## Properties of a $\tau$ -structure

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A **property** of  $\tau$ -structures is a class  $\mathcal{P}$  of  $\tau$ -structures **closed under  $\tau$ -isomorphisms**.

### Examples:

- All *finite*  $\tau$ -structures.
- All  $\{R_{2,0}\}$ -structures where  $R_{2,0}$  is interpreted as a linear order.
- All finite 3-dimensional matchings *3DM*, i.e. all  $\{R_{3,0}\}$ -structures with universe  $A$  where the interpretation of  $R_{3,0}$  contains a subset  $M \subseteq A^3$  such that no two triples of  $M$  agree in any coordinate.
- All binary words which are palindroms.

We say a  $\tau$ -structure  $\mathcal{A}$  **has property  $\mathcal{P}$**  iff  $\mathcal{A} \in \mathcal{P}$ .

## First Order Logic FOL

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We now assume our vocabularies are **one-sorted** with sort symbol  $V$ .

We define the set of formulas  $\text{FOL}(\tau)$ :

**Variables:**  $u, v, w, \dots$  ranging over elements of the interpretation of  $V$ .

**Terms:** Variables and constant symbols in  $\tau$  are  $\tau$ -terms.

**Atomic formulas:** For each  $R_{m,j} \in \tau$  and  $\tau$ -terms  $t_1, \dots, t_m$  the expressions  $R_{m,j}(t_1, \dots, t_m)$ ,  $t_1 = t_2$  are atomic formulas in  $\text{FOL}(\tau)$ .

**Boolean connectives:** If  $\phi$  and  $\psi$  are in  $\text{FOL}(\tau)$ , so are  $\phi \wedge \psi$ ,  $\phi \vee \psi$ ,  $\phi \Rightarrow \psi$  and  $\neg\phi$ .

**Quantifiers:** If  $\phi$  is in  $\text{FOL}(\tau)$  and  $v$  is a variable, then  $\exists v\phi$  and  $\forall v\phi$  are in  $\text{FOL}(\tau)$ .

## Second Order Logic SOL

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We now define  $\text{SOL}(\tau)$ , the set of SOL-formulas for a vocabulary  $\tau$ :

**FOL** :  $\text{FOL}(\tau) \subseteq \text{SOL}(\tau)$  and  $\text{SOL}(\tau)$  is closed under boolean connectives and first order quantification.

**Second order variables:** For each  $m, j \in \mathbb{N} - \{0\}$  we have second order variables  $X_{m,j}$  of arity  $m$ .  
For each  $X_{m,j}$  a second order variable, and  $\tau$ -terms  $t_1, \dots, t_m$  the expression  $X_{m,j}(t_1, \dots, t_m)$ , is an atomic formulas in  $\text{SOL}(\tau)$ .

**Second order quantification:** If  $\phi \in \text{SOL}(\tau)$  so are  $\forall X_{m,j} \phi$  and  $\exists X_{m,j} \phi$ .

**Monadic Second Order formulas  $\text{MSOL}(\tau)$**  are those where for the arity  $m$  of the second order variables we have  $m = 1$ .

Analogously,  $\text{SOL}^n(\tau)$  is obtained by restricting the arity  $m$  of the second order variables to  $m \leq n$ .

## Outline of the course

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**LECTURE 00:** Second Order Logic (SOL) and its fragments (Background, not lectured)  
LOGICS (14 slides)

**LECTURE 01:** Friday, Oct 10, 2014, 14:00-15:40, Prague Lecture 1,  
A landscape of graph parameters and graph polynomials. Comparing graph parameters.  
Towards a general theory.  
(90 minutes, 90 slides with skip-options)

**LECTURE 02:** Thursday, Oct 16, 2014, 12:20-14:00 Prague Lecture 2,  
Why is the chromatic polynomial a polynomial? Where to graph polynomial occur  
naturally? Definability of graph properties and graph polynomials in fragment of Second  
Order Logic.  
(90 minutes, ca. 99 slides with skip options)

**LECTURE 03:** Thursday, Oct 16, 2014, 14:30-16:00 Prague Lecture 3,  
Connection matrices for graph parameters. When do connection matrices of graph  
parameters have finite rank? Connection matrices for graph parameters definable in  
fragments of Second Order Logic. The finite rank theorem. Using connections matrices  
to prove non-definability.  
(90 minutes, ca. 55 slides with skip options)

Further links to the literature.

[File:p-overview](#)

## Further links

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- [**arXiv** ] J.A. Makowsky's Graph Polynomial **Go to Homepage** at <http://www.cs.technion.ac.il/~janos/RESEARCH/gp-homepage.html>
- [**KMR 2013** ] J. A. Makowsky T. Kotek and E. V. Ravve, **A computational framework for the study of partition functions and graph polynomials**. In Proceedings of the 12th Asian Logic Conference '11, pages 210-230, 2013. **download** at <http://www.cs.technion.ac.il/~janos/RESEARCH/alcpaper.pdf>
- [**GKM 2012** ] B. Godlin, E. Katz and J. A. Makowsky, **Graph Polynomials: From Recursive Definitions to Subset Expansion Formulas**. J. Log. Comput. 22(2): 237-265 (2012) **download** at <http://www.cs.technion.ac.il/~janos/RESEARCH/GodlinKatzMakowsky.pdf>
- [**M 2008** ] J.A. Makowsky, **From a Zoo to a Zoology: Towards a general theory of graph polynomials**, Theory of Computing Systems, 2008. **download** at <http://dx.doi.org/10.1007/s00224-007-9022-9>

More links

[File:p-overview](#)

## Further links, II

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**[arXiv ]** J.A. Makowsky's [papers](http://arxiv.org/find/all/1/au:+Makowsky/0/1/0/all/0/1?per_page=100) at [http://arxiv.org/find/all/1/au:+Makowsky/0/1/0/all/0/1?per\\_page=100](http://arxiv.org/find/all/1/au:+Makowsky/0/1/0/all/0/1?per_page=100) on [arXiv](#).

**[dblp ]** J.A. Makowsky's [papers](http://www.informatik.uni-trier.de/~ley/pers/hd/m/Makowsky:Johann_A=.html) at [http://www.informatik.uni-trier.de/~ley/pers/hd/m/Makowsky:Johann\\_A=.html](http://www.informatik.uni-trier.de/~ley/pers/hd/m/Makowsky:Johann_A=.html) on [DBLP](#).

**[google ]** J.A. Makowsky's [papers](http://scholar.google.co.il/citations?hl=en&user=ooNKL6UAAAAJ&pagesize=100&view_op=list_works) at [http://scholar.google.co.il/citations?hl=en&user=ooNKL6UAAAAJ&pagesize=100&view\\_op=list\\_works](http://scholar.google.co.il/citations?hl=en&user=ooNKL6UAAAAJ&pagesize=100&view_op=list_works) at [scholar.google](#).

**[Course notes ]** J.A. Makowsky's [Course notes](#).

**[PhD Theses ]** [PhD Theses](#) on graph polynomials (a selection)

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## Further links: Course notes

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Slides of courses on graph polynomials and related topics:

**Technion 2005/6** [Lecture notes](#) of **Advanced Topics in Computer Science (238900)**

**Technion 2009/10** [Lecture notes](#) of **Advanced Topics in Computer Science (236605)**

**Vienna 2014** [Lecture notes](#) of **EMCL Lecture 2014: Graph polynomials**

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## Further links: PhD Theses

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PhD Theses on graph polynomials and related topics:

- I. Averbuch** [PhD Thesis](#) (Technion 2011): [Completeness and Universality Properties of Graph Invariants and Graph Polynomials](#)
- T. Kotek** [PhD Thesis](#) (Technion 2012): [Definability of combinatorial functions](#)
- M. Trinks** [PhD Thesis](#) (TU Freiberg 2012): [Graph Polynomials and Their Representations](#)

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