Topics in Automated Theorem Proving

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Lecture 7:

The Decidability of Elementary Geometry

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Tarski's Theorem of 1931 and 1951

Elementary Euclidean Geometry is decidable

This needs clarifications:

- What is elementary Geometry?
- What do we mean by decidable?

Apology

- There are no new technical results in this talk.
- I just report on what I learned when I reviewed the question while prepariung a course in 2003.
- But I would like to draw attention to Ziegler's results and their significance for the question.

  They have been widely overlooked, due to the fact that they were published in German in a Swiss-French periodical in 1982.
- I also offer a comprehensive view, both Algebra-Geometrical and Algorithmic.

The Classics

Euclides: *Elements of Geometry*

The most influental mathematical text ever written.

Latin versions: Peletier, 1557; F. Commandino, 1572; C. Clavius, 1574.

Italian version: F. Commandino, 1575

French version: F. Peyrard, 1804

English versions: Simson, 1756; Playfair 1795; Heath, 1926

Descartes: *Discours sur la méthode*,
with an appendix *La Géométrie* 1637 and 1664.

Euclides Danicus: Georg Mohr (1640-1697), published in 1672

Hilbert: *Grundlagen der Geometrie*, 1899 ff.
David Hilbert (later editions with P.Bernays),
English version by Leo Unger, 1971

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Elimination of quantifiers in real closed fields

Tarski really proved

The first order theory of the reals,
with addition, multiplication and order,
admits elimination of quantifiers
by a computable procedure.

The computable procedure in his proof is not elementary.

The model of computation allows the use of real numbers.

Decidability of the first order theory of Real Closed Fields

The elimination of quantifiers allows us
to reduce first order formulas
with addition, multiplication, and order
to quantifier free formulas.

This leaves us with
a boolean combination of polynomial (in)equalities.

If the coeffiecients of the polynomials are rational, the truth of such formulas can be algorithmically decided.

The computational models

- In the Turing model

  Real numbers must be finitely presentable

- In the Blum-Shub-Smale Model (BSS)

  Real numbers are black boxes with decidable equality

Note that the BSS Model has many precursors between 1950-1970.

J. Shepherdson and H. Sturgis, Y. Ershov, J. McCarthy,
E. Engeler, V. Harnik, G. Herman and S. Isard, H. Friedman

What are geometric objects?

The basic objects are

Points $(P_1, P_2, \ldots)$, lines $(l_1, l_2, \ldots)$, planes, hyperplanes, ...

with the relationships of

Incidence: A point is on a line $P \in l$, in a plane, ...

Equidistance: $Eq(P_1, P_2, Q_1, Q_2)$,

Orthogonality: $Or(P_1, P_2, Q_1, Q_2)$ or $Or(l_1, l_2)$,

Congruence of angles: $An(P_1, P_2, P_3, Q_1, Q_2, Q_3)$

Betweenness: $Be(P_1, P_2, P_3)$

What is a geometric statement?

- Atomic statements: Assertion of a relationship
- Basic statement: Boolean combination of atomic statements
- Classical theorems (in Euclid): From a given basic statement an atomic statement follows

Example: The three bisectors of a triangle meet in one point.

What is a geometric construction?

Ruler-1: Intersection of two lines: $P := l_1 \times l_2$.

Ruler-2: Line through two points: $l := Lin(P_1 P_2)$.

Ruler-3: Parallel line: $l_1 := Par(A, l)$, for $A \notin l$.

Ruler-4: Orthogonal line: $l_1 := Orth(A, l)$, for $A \notin l$.

Compass-1: Circle with center A and radius B_1, B_2: $P_1 \in Circ(A, B_1, B_2)$.

Compass-2: Intersection of two circles.

Example: The three bisectors of a triangle meet in one point.
Construct this point.

Descriptions of constructions (straight line programs) are classical theorems.

Beyond Ruler and Compass, I

For the three dimesional constructions in Euclid, he uses two more constructs:

3D-Ruler: Draw a plane through three given points.

3D-Compass: Rotate a semcircle around its axis.

Beyond Ruler and Compass, II

Other constructions in the plane use

Marked Ruler: Given two lines l, m and a distance d and a point O one can draw a line l' with three points $O, A, B \in l'$ such that $A \in l, B \in m$ and $AB = d$.

Auxiliary curves: Parabola or quadratrix.

The Conchoid of Nicodemes: Given a distance d, a line l and a points Q_1, Q_2, the concoide is the locus of all points P such that $line(Q_1, P)$ cuts l in Q_2 and $PQ_2 = d$.

Ellipse construction with a rope: Fix a rope of length d at two points A, B with $AB \leq d$ and use it two draw the locus of all P $AP + BP = d$.

All these constructions are describable in Cartesian coordinates by algebraic equations.

Origami Geometry

The Japanese developed the art of paper folding ORIGAMI (link to wikipedia).

- Origami constructions. (web-link)
- First order structures for Origami, (web-link
- We shall discuss this later, if time permits.

Construction vs description

Sums and product of distances are describable.

Do there **exist** points A_1, A_2, B_1, B_2 such that their distances $x = A_1A_2$, $y = B_1B_2$ satisfy the equation

$$y^3 = 2x^3?$$

Can we **construct** points A_1, A_2, B_1, B_2 such that their distances $x = A_1A_2$, $y = B_1B_2$ satisfy the equation

$$y^3 = 2x^3?$$

Classically,

existence means **constructible**.

Existential quantifiers

In modern terms we would say

There are points A_1, A_2, B_1, B_2 such that their distances $x = A_1A_2, y = B_1B_2$ satisfy the equation $y^3 = 2x^3$,

but **Galois and Abel** provided the techniques to show that these points area**in general not constructible**.

We assume actually, that **any describable situation consistent with the axioms of geometry**, can be realized.

A problem of data modeling

Geometry is the oldest problem of **data modeling**.

The objects of geometry **are** points, lines, planes, . . .

Are they?

They are thought of as points, lines, planes . . .

Points are **modeled as** vectors over a **number domain**, like the field of the rationals, reals, algebraic reals, ...

They are modeled as **sets** . . .

The engineering point of view

The geometry of

- civil engineering and architecture
- classical mechanics
- classical optics

is properly modeled as the

Analytical geometry over the reals

as a complete ordered field.

The construction point of view

The geometry of construction

- by ruler alone is caputered by the analytic geometry of ordered fields.
- by ruler and angle-bisector (or dividers) is caputered by the analytic geometry of ordered pythagorean fields where every sum of two squares is a square.
- by ruler and compass is caputered by the analytic geometry of ordered euclidean fields where every positive number is a square.
- by marked ruler is caputered by the analytic geometry of ordered Vieta fields where every polynomial of degree 4 which has a real root have roots.
  This can be formalized in first order logic.
- The algebraic characterization of the constructions by marked ruler and compass is **open**.

There are more geometries in this world

- Hyperbolic geometries

  can also be described with fields

- Finite geometries

- Geometries on surfaces

- Geometries from modern physics

Axioms of Plane Geometry

- Incidence axioms
- Parallel axiom
- Congruence axioms (equidistance)
- Congruence axioms (equiangularity)
- Congruence axioms (orthogonality)
- Axioms of infinity
- Axioms of betweenness
- Axioms of continuity

Incidence axioms

I-1: For any two distinct points A, B there is a unique line l with $A \in l$ and $B \in l$.

I-2: Every line contains at least two distinct points.

I-3: There exists three distinct points A, B, C such that no line l contains all of them.

Parallel axiom

We define: $Par(l_1, l_2)$ or $l_1 \parallel l_2$
if l_1 and l_2 have no point in common.

Parallel Axiom: For each point A and each line l there is at most one line l' with $l \parallel l'$ and $A \in l'$.

Axioms of betweenness

B-1: If $Be(A, B, C)$ then there is l with $A, B, C \in l$.

B-2: For every A, B there is C with $Be(A, B, C)$.

B-3: For each $A, B, C \in l$ exactly one point is between the others.

B-4: (Pasch) Given A, B, C and l in general position,
(the three points are not on one line, none of the points on l),
if $Be(A, D, B)$ there is $D' \in l$ with $Be(A, D', C)$ or $Be(B, D', C)$.

Congruence axioms: Equidistance

We write for $Eq(A,B,C,D)$ the usual $AB \cong CD$.

C-0: $AB \cong AB \cong BA$.

C-1: Given A,B,C,C', l with $C,C' \in l$
there is a unique $D \in l$ with
$AB \cong CD$ and $B(C,C',D)$ or $B(C,D,C')$.

C-2: If $AB \cong CD$ and $AB \cong EF$ then $CD \cong EF$.

C-3: (Addition)
Given A,B,C,D,E,F with $Be(A,B,C)$ and $Be(D,E,F)$,
if $AB \cong DE$ and $BC \cong EF$, then $AC \cong DF$.

C-1 and C-3 use the betweenness relation Be.

Congruence axioms: Equiangularity

We denote by $\vec{AB}$ the directed ray from A to B.
We denote by $\angle(ABC)$ the angle between $\vec{AB}$ and $\vec{BC}$.
$\angle(ABC) \cong \angle(A'B'C')$ the congruence of angles.

C-4: Given rays $\vec{AB}$, $\vec{AC}$ and $\vec{DE}$

there is a unqique ray $\vec{DF}$ with $\angle(BAC) \cong \angle(EDF)$.

C-5: Congruence of angles is an equivalence relation.

C-6: (Side-Angle-Side)

Given two triangles ABC and $A'B'C'$

with $AB \cong A'B'$, $AC \cong A'C'$ and $\angle BAC \cong \angle B'A'C'$

then $BC \cong B'C'$, $\angle ABC \cong \angle A'B'C'$ and $\angle ACB \cong \angle A'C'B'$.

Congruence axioms: Orthogonality

We denote by $l_1 \perp l_2$ the orthogonality of two lines $Or(l_1, l_2)$.
We call a line l **isotropic** if $l \perp l$. This is apriori possible.

O-1: $l_1 \perp l_2$ iff $l_2 \perp l_1$.

O-2: Given O and l_1, there exists exactl one line l_2 with $l_1 \perp l_2$ and $O \in l_2$.

O-3: $l_1 \perp l_2$ iff $l_1 \perp l_3$ the $l_2 \parallel l_3$.

O-4: For every O there is an l with $O \in l$ and $l \not\perp l$.

O-5: The three heights of a triangle intersect in one point.

Axioms of Desargues and of infinity

Infinity: Given distinct A, B, C and l with $A \in l, B, C \notin l$

we define $A_1 = Par(AB, C) \times l$, and inductively,

$A_{n+1} = Par(A_n B, C) \times l$.

Then all the A_i are distinct.

Desargues-1: If AA', BB', CC' intersect in one point or are all parallel, and $AB \parallel A'B'$ and $AC \parallel A'C'$ then $BC \parallel B'C'$.

Desargues-2: If $AB \parallel A'B'$, $AC \parallel A'C'$ and $BC \parallel B'C'$ then AA', BB', CC' are all parallel.

Axiom of Symmetric Axis and Transposition

Axiom of Symmetric Axis Any two intersecting non-isotropic lines have a symmetric axis.

Axiom of Transposition Let l, l' be two non-isotropic lines with $A, O, B \in l$, $AO \cong OB$ and $O' \in l'$ there are exactly two points $A', B' \in l'$ such that $AB \cong A'B' \cong B'A'$ and $A'O' \cong O'B'$.

The two axioms are equivalent in geometries satisfying the Incidence, Parallel, Desargues and Orthogonality axioms together with the axiom of infinity.

Axiom E

Axiom E

Given two circles Γ, Δ such that Γ contains at least one point inside , and one point outside Δ, then $\Gamma \cap \Delta \neq \emptyset$.

Models of geometry for incidence relation only (τ_0)

The axioms I-1, I-2, I-3, Parallel and Infinity use only the incidence relation.

Models of **affine Geometry** are exactly those which satisfy the above and

Pappus

Given two lines l, l' and points $A, B, C \in l$ and $A', B', C' \in l'$ such that $AC' \parallel A'C$ and $BC' \parallel B'C$. Then also $AB' \parallel A'B$.

Models of geometry using incidence, betweenness, equidistance and equiangularity ($\tau_{Hilbert}$).

Hilbert Plane:

Axioms I-1, ..., I-3,
B-1, ... , B-4,
C-1, ..., C-6.

Euclidean Plane:

Hilbert Plane with
Parallel Axiom and Axiom E.

Models of geometry using incidence, equidistance and orthogonality (τ_{Wu}).

Orthoganal Wu Plane:

I-1, ..., I-3,
O-1, ..., O-5,
Parallel Axiom, Infinity, Desargues,

Metric Wu Plane:

Orthogonal Wu Plane satisfying additionally
the axioms Symmetric Axis
(or equivalently) Transposition.

Cartesian coordinates over arbitrary fields.

Given a field $\mathfrak{F}$ we define the Cartesian Plane by taking points as pairs of coordinates and lines as solution of linear equations.

We can also introduce a norm, and angles (in the standard way).

Let us denote the corresponding structure with the incidence, equidistance, equiangularity, orthogonality relations as $\Pi_{\mathfrak{F}}$.

Theorem:
If $\mathfrak{F}$ is any field, $\Pi_{\mathfrak{F}}$ satisfies the Incidence and the Parallel Axioms and the Pappus Axiom.

If $\mathfrak{F}$ is additionally of characteristic 0, $\Pi_{\mathfrak{F}}$ also satisfies the axiom of infinity, hence is a model of Affine Geometry.

Number systems in Affine Geometry.

In (sufficiently axiomatized) models of Plane Geometry Π there is a standard way of adding and multiplying distances which gives rise to a commutative ring which we denote by $\mathfrak{R}_\Pi$.

Theorem:(Artin)
If Π is a model of Affine Geometry, $\mathfrak{R}_\Pi$ is a field of characteristic 0.

A field has the Hilbert (Pythagorean) Property if square roots of sums of squares exist, i.e.

$$\forall z(\exists x, y(z = x^2 + y^2) \to \exists u(u^2 = z))$$

Theorem:(Wu)
If $\mathfrak{F}$ is a Hilbert field of characteristic 0, then $\Pi_{\mathfrak{F}}$ is a Metric Wu Plane.

If Π is a Metric Wu Plane, $\mathfrak{R}_\Pi$ is a Hilbert field of characteristic 0.

Geometries and ordered fields.

Theorem:
Given a Hilbert Plane Π which satisfies the Parallel Axiom, then $\mathfrak{F}_{\Pi}$ is a field of characteristic 0 which can be uniquely ordered to be an ordered field.

Conversely, in any ordered field $\mathfrak{F}$ which has the Pythagorean Property (a Pythagorean field), its Plane $\Pi_{\mathfrak{F}}$ is a Hilbert Plane which satisfies the Parallel Axiom.

There are various similar theorems.

Tarski's dimensionfree geometry, I

Objects: Points

Relations:

Congruence $D(a,b,c,d)$ or $ab \equiv cd$, the distance between a and b is the same as between c and d (including for $a = b$).

Betweenness $B(a,b,c)$, the three points a,b,c are colinear and b is between a and c.

With axioms A1-A7, A10 dimension free,

Axiom A8 says the dimension is at least 2,
Axiom A9 says the dimension is at most 2.

and A11 is the axiom of continuity, A11' its first order version.

CA is the circle axiom (similar to axiom E of Hilbert).

Tarski's Axioms, I

A1: $ab \equiv ba$.

A2: If $ab \equiv pq$ and $ab \equiv rs$ then also $pq \equiv rs$.

A3: (Identity for congruence of segments)
If $ab \equiv cc$ then $a = b$.

A4: (Axiom of transfer of concruent segments)
For every a, b, c, q there is an x with $B(qax)$ and $ax \equiv bc$.

A5: (Axiom of five segments)
If $a \neq b$, $B(a, b, c)$, $B(a', b', c')$, and
$ab \equiv a'b'$, $bc \equiv b'c'$, $ad \equiv a'd'$, $bd \equiv b'd'$,
then $cd \equiv c'd'$.

Tarski's Axioms, II

A6: (Identity for betweenness)
If $B(aba)$ then $a = b$.

A7: (Axiom of Pasch, inner version)
If $B(a, p, c)$ and $B(b, q, c)$ there is an x with $B(p, x, b)$ and $B(q, x, a)$.

A8: (Lower dimension axiom)
There are three points a, b, c with $\neg B(a, b, c)$ and $\neg B(b, c, a)$ and $\neg B(c, a, b)$.

A9: (Upper dimension axiom)
If $p \neq q$ and $ap \equiv aq$ and $bp \equiv bq$ and $cp \equiv cq$
then $B(a, b, c)$ or $B(b, c, a)$ or $B(c, a, b)$.

A10: (Euclidean axiom)
If $B(a, d, t)$ and $B(b, d, c)$ and $a \neq d$
there are x, y with $B(a, b, x)$ and $B(a, c, y)$ and $B(x, t, y)$.

Tarski's Axioms, III

A11: (Continuity)
For any two sets of points X, Y, if there is a such that for all $x \in X$ and $y \in Y$ one has $B(a, x, y)$
then there is a b such that for all $x \in X$ and $y \in Y$ one has $B(x, b, y)$.

A11': (Schema of first order continuity)
For any two definable sets of points $X = \phi(x), Y = \psi(y)$, if there is a such that for all x, y with $\phi(x)$ and $\psi(y)$ one has $B(a, x, y)$
then there is a b such that for all x, y with $\phi(x)$ and $\psi(y)$ one has $B(x, b, y)$.

CA: (Circle axiom)
If $B(c, q, p)$, $B(c, q, r)$, $ca \equiv cq$ and $cb \equiv cr$ then there is an x with $cx \equiv cp$ $B(a, x, b)$.

Tarski's dimensionfree geometry, II

Let F be an ordered field. With the traditional definitions in F^n we have

(i) F^n is a model of A1-A3, A5-A7 and A10.

(ii) F^n is a model of A4 iff F is pythagorean.

(iii) F^2 satisfies A8 and A9.

(iv) For $n \geq 1$, F^n satisfies A11 (continuity) iff $F = \mathbb{R}$.

(v) For $n \geq 2$, F^n satisfies A11' (continuity) iff F is real closed.

(vi) For $n \geq 2$, F^n satisfies CA iff F is an euclidean field.

The converse are also true.

Asking again: What is Elementary Geometry?

Let $\mathcal{L}$ be a subset of all first order sentences in the language of geometry (τ_0, $\tau_{Hilbert}$ or τ_{Wu}).

Let Γ be a set of geometrical axioms.

We call the set

$$Th_{\mathcal{L}}(\Gamma) = \{\phi \in \mathcal{L} : \phi \text{ true in all models of } \Gamma\}$$

the $\mathcal{L}$-Theory of Γ.

Problem: For which Γ and $\mathcal{L}$ is $Th_{\mathcal{L}}(\Gamma)$ decidable?

By the various correspondences this reduces to:

Problem: What theories of (ordered) fields are decidable?

Afred Tarski and Julia Robinson

Theorem:(A. Tarski, 1935, 1951)
The full first order theory of algebraically closed fields and real closed fields both allows elimination of quantifiers and have decidable term equality, hence are decidable.

Theorem:(J. Robinson, 1949)
Both the full first order theory of fields and of ordered fields are undecidable.

Question: Are the full first order theories decidable for

- Affine Geometry,
- Hilbert or Euclidean Plane,
- Orthogonal or Metric Wu Plane?

In particular is the theory of pythagorean or euclidean fields decidable?
(Tarski in 1959 conjectured that no.)

More undecidability results

Theorem:(J. Robinson, 1949)
The complete first order theory of the rationals as a field is undecidable.

Theorem:(A. Mal'cev, 1961)
The full first order theory of the the theory of rings is undecidable.

Ziegler's Theorem

Theorem:(M. Ziegler, 1982)

Let T be a finite theory consistent with the theory of algebraically closed fields of characteristic 0 or with the theory of (real closed) fields, then T is undecidable.

In particular the full first order theories of all our Geometries above are undecidable.

Rautenberg and Hauschild - a Cold War Tale

1973: W. Rautenberg and K. Hauschild in East Berlin announce their result, that the theory of pythagorean fields is undecidable.

1973 Rautenberg leaves East Berlin in an adventurous and illegal way to the West and visits Berkeley. Taking merit for the result he becomes Professor in West Berlin.

1974: The result is published in Fundamenta Mathematicae without Rautenberg's name in the paper (but it does appear on the top of even numbered pages).

1977: K. Hauschild publishes an Addendum to the paper in Fundamenta Mathematicae.

1979: H. Ficht in his M.Sc. thesis written under A. Prestel finds an irreparable mistake in the proof. M. Ziegler is a co-examinor.

1980: Martin Ziegler presents his alternative and more general proof.

How to justify real closure?

Continuity:

In the Hilbert plane one can formulate (in second order logic) the principle of continuity (Dedekind cuts).

Tarski introduces (artificially) the first order scheme for definable Dedekind cuts (FOL-Continuity).

Theorem: In any Hilbert plane Π which satisfies additionally the Parallel Axiom the field $\mathfrak{F}_{\Pi}$ is real closed iff Π satisfies FOL-Continuity.

How to justify algebraic closure?

Enough roots:

Given a distance AB and a polynomial with integer coefficients $p(x)$ we can find to points C, D such that for the distance CD we have $CD^2 \cong p(AB)^2$.

We can formulate a converse (EnoughRoots):
Given a distance CB and a polynomial with integer coefficients $p(x)$ we can find to points A, B such that for the distance AB we have $CD^2 \cong p(AB)^2$.

Theorem: In any Orthogonal Wu Plane Π the field $\mathfrak{F}_{\Pi}$ is algebraically closed iff Π satisfies EnoughRoots.
Similarly for Affine Geoemtry and for the Euclidean Plane.

Verification of Geometric Constructions
High School Geometry

In text book problems in Geometry we are a given a construction of points $P_1, P_2, \ldots, P_n$ and lines $l_1, l_2, \ldots, l_m$ using **ruler and compass**. The theorem then asserts or forbids that a subset of points either meet, are colinear or cocircular, a subset of lines either meet, are parallel or perpendicular, or a subset of pairs of points are pairwise equidistant.

Translating this into the language of (ordered) fields we get a formula of the form

$$\forall \bar{x} \left(\left(\bigwedge_{i \in I} f_i(\bar{x}) = 0 \wedge \bigwedge_{j \in J} h_j(\bar{x}) \neq 0 \right) \to g(\bar{x}) = 0 \right)$$

Here the f_i, h_j, g are polynomials of degree 2. In particular, the statement is of the form $\forall \bar{x} \Phi(\bar{x})$, with Φ quantifier free.

This remains true if we allow also constructions with **marked ruler** which allows us to trisect angles.

The universal theory of Affine Geometry

Theorem: (Schur)
For any model of Affine Geoemtry Π the ring $\mathfrak{F}_{\Pi}$ is a commutative field of characteristic 0.
Conversely, for a commutative field of characteristic 0, $\mathfrak{F}$, the Geometry $\Pi_{\mathfrak{F}}$ is a model of Affine Geometry.

Theorem: Let T be a set of τ_{Wu}-sentences ($\tau_{Hilbert}$-sentences) and let ϕ be a universal τ_{Wu}-sentence ($\tau_{Hilbert}$-sentence).

(i) If T has an algebraic closed field as model, then

$$T \vdash \phi \text{ iff } ACF_0 \vdash \phi.$$

(ii) If T has a real closed field as model, then

$$T \vdash \phi \text{ iff } RCF \vdash \phi.$$

In particlar, in both cases the universal theory of the Geometry derived from T is decidable.

Conclusions, I

- Quantified first order properties of
  the **Real (Euclidean) Plane** are decidable.

- Quantified first order properties true in all
  Affine Planes
  (Hilbert, Euclidean, Orthogonal and Metric Wu Planes)
  are undecidable.

- Universal statements true in
  all Geometries above are decidable.

  What about $\forall\exists$ statements?

Complexity

- Tarski's proof gives a procedure which is not bounded by any finite iteration of the exponential function.

- G. Collins (1975) gives a different approach to quantifier elimination in RCF, called **Cylindrical decomposition**, which has a doubly exponential upper bound.

- D. Grigoriev 1984 (also with A. Chistov and N. Vorobjov 1988) gives a better algorithm which takes into account the quantifier alternations. For universal formulas it is simply exponential. (similar results follow later but independently by J. Heintz, 1990, and J. Renegar, 1992)

- Lower bounds for quantifier elimination in RCF are due to M. Ben-Or, D. Kozen and J. Reif, 1986.

- For universal formulas in ACF_0, W. Wu devised a method based on ideas due to R.Ritt, 1984, and which is similar to methods using **Gröbner bases** for ideals of polynomials, due to B. Buchberger, 1970.

Conclusions, II

Tarski's work initiated interest in computational geometry. The following are highly challenging and demanding further avenues:

- Computational algebraic geometry (both real and complex).
- Computational geometry.
- Algorithmic topology and knot theory.

Tank you for your attention.