A landscape of graph polynomials

Graph isomorphims

Let $\mathcal{DG}$ be the class of finite graphs $\langle V(G), E(G) \rangle$ where $V + V(G)$ is a finite set and $E + E(G) \subseteq V^2$ is a set of (directed edges). $G \in \mathcal{DG}$ is called a directed graph. $\mathcal{G}$ be the class of finite graphs, i.e. where E is symmetric.

For $G_1, G_2 \in \mathcal{DG}$ $f : G_1 \rightarrow G_2$ is an **isomorphisms** if

(i) f is a bijection, and

(ii) For $u, v \in V(G_1)$ we have
$(u, v) \in E(G_1)$ iff $(f(u), f(v)) \in E(G_2)$.

G_1 **and** g_2 **are isomorphic**, denoted by $G_1 \simeq G_2$,
if there is an isomorphism $f : G_1 \rightarrow G_2$.

Rings $\mathcal{R}$

Let $\mathcal{R}$ a ring.

- $\mathcal{R} = \mathcal{B}_2$ the two element bolean ring.
- $\mathcal{R} = \mathbb{Z}_2$ the two element field.
- $\mathcal{R} = \mathbb{Z}$, the ring of integers.
- $\mathcal{R} = \mathbb{Z}[X]$, the polynomial ring over the integers with one indeterminate.
- $\mathcal{R} = \mathbb{Z}[X_1, \ldots, X_k]$, the polynomial ring over the integers with k indeterminates.
- $\mathcal{R} = \mathbb{R}$, the ring of real numbers.

Definition 1 *Graph invariants over a ring $\mathcal{R}$*

Let $\mathcal{R}$ a ring, $\mathcal{G}$ the class of finite graphs.
A function

$$f : \mathcal{G} \to \mathcal{R}$$

is a graph invariant if for any two isomorphic graphs G_1, G_2 we have $f(G_1) = f(G_2)$.

Example 2 *Boolean graph invariants*

Here the ring is $\mathcal{B}_2$,
or any ring $\mathcal{R}$, but the values of the invariant are either 0 or 1.

- Connectedness
- Regular, or regular of degree r.
- Any First Order expressible graph property.
- Any Second Order expressible graph property.
- Belonging to any specific class of graph closed under isomorphisms.
- There are continuum many boolean graph invariants.

Example 3 *Numeric graph invariants*

Here the ring is $\mathbb{Z}$.

- The cardinality of $V(G)$ or $E(G)$.
- The number of connected components of G, usually denoted by $k(G)$.
- The coloring number of G.
- The size of the maximal clique (independent set).
- The diameter of G.
- The radius of G.
- The minimum length of a cycle in G, if it exists, called the girth of the graph G.

Example 4 *Graph polynomials*

Here the ring is $\mathbb{Z}[X]$.

The graph polynomial $p(G, X)$ gives for each value of X a graph invariant, hence it encodes a possibly infinite family of graph invariants.

The study of graph polynomials has a long history concentrating on particular polynomials.

The **classic** and very readable book is:

- Norman Biggs
  Algebraic Graph Theory
  Cambridge University Press
  1974 (2nd edition 1993)

Example 5 *The chromatic polynomial*

- Let $\chi(G, X)$ denote the number of vertex colorings of G with X colors.
  We shall prove that $\chi(G, X)$ is a polynomial in X,
  called the **chromatic polynomial of** G.

The chromatic polynomial was first introduced by G.D. Birkhoff in 1912.

It led to a very rich theory, although it was introduced
in a (failed) attempt to prove the 4-color conjecture.

The most comprehensive monograph about the chromatic polynomial is

- F.M. Dong, K.M. Koh and K.L. Teo
  Chromatic polynomials and chromaticity of graphs
  World Scientific, 2005

What can we do with a graph polynomial?

- Study its zeros.
- Interpret its coefficients in various normal forms.
- Interpret its evaluations.
- Study graphs uniquely determined by the polynomial.
- Study graph classes having the same graph polynomial.
- Study the strength of the graph invariant.

Digression 1:

Typical theorems about the chromatic polynomial

Theorem 1 (G. Birkhoff, 1912)

$\chi(G, X)$ is indeed a polynomial in X of degree $\mid V(G) \mid$.

Proof Let $e = (a, b)$ be an edge of the graph G. $G - e$ and G/e are obtained from G by deleting, respectively contracting the edge e.
We use induction over $E(G)$.

- First we observe that for disjoint unions $G = G_1 \sqcup G_2$ we have $\chi(G, X) = \chi(G_1, X) \cdot \chi(G_2, X)$.
- For n isolated points $\bar{K_n}$ we have $\chi(\bar{K_n}, X) = X^n$.
- $\chi_{a \neq b}(G, X)$ is the number of X-colorings of G with a and b having different colors.
- $\chi_{a=b}(G, X)$ is the number of X-colorings of G with a and b having the same color.
- $\chi(G - e, X) = \chi_{a \neq b}(G - e, X) + \chi_{a=b}(G - e, X) = \chi(G, X) + \chi(G/e, X)$
- $\chi(G, X) = \chi(G - e, X) - \chi(G/e, X)$ Q.E.D.

Normal forms of $\chi(G, X)$, I

As $\chi(G, X)$ is a polynomial we can write it as

$$\chi(G, X) = \sum_{i}^{|V(G)|} b_i(G) X^i$$

For the disjoint union we noted that

Proposition 2

$$\chi(G_1 \sqcup G_2, X) = \chi(G_1, X) \cdot \chi(G_2, X).$$

Normal forms of $\chi(G, X)$, II

We define $X_{(i)} = X \cdot (X-1) \cdot \ldots \cdot (X-i+1)$.
We write

$$\chi(G, X) = \sum_{i}^{|V(G)|} c_i(G) X_{(i)}$$

We define a an operation $\circ$ on the $X_{(i)}$ by $X_{(i)} \circ X_{(j)} = X_{(i+j)}$ and extend it naturally to polynomials in $X_{(i)}$.

The join of two graphs G_1, G_2, $G_1 + G_2$,
is obtained by taking the disjoint union and
adding all the edges between $V(G_1)$ and $V(G_2)$.

Theorem 3

$$\chi(G_1 + G_2, X) = \left(\sum_{i}^{|V(G)|} c_i(G_1) X_{(i)} \circ \sum_{i}^{|V(G)|} c_i(G_2) X_{(i)} \right)$$

Trees and tree-width

- For trees T with n vertices we have $\chi(T,X) = X \cdot (X-1)^{n-1}$.
  I particular, any two trees on n vertices have
  the same chromatic polynomial.

- (R. Read, 1968)
  Conversely, for G a simple graph, if $\chi(G,X) = X \cdot (X-1)^{n-1}$, then G is a tree.

- (C. Thomassen, 1997)
  If G has tree-width $k \geq 2$ then for every real number $a > k$ we have $\chi(G,a) \neq 0$.

- (B. Courcelle, J.A. Makowsky, U. Rotics, 2000)
  For graphs G with tree-width at most k the polynomial $\chi(G,X)$ can be computed in polynomial time.

- (J.A. Makowsky, U. Rotics, 2005)
  For graphs G with clique-width at most k the polynomial $\chi(G,X)$ can be computed in polynomial time.

Planar graphs and the chromatic polynomial.

Theorem 4 (P.J. Heawood, 1890)
Every planar graph is 5-colorable.
$\chi(G,5) \neq 0$ *for* G *planar.*

Theorem 5 (G. Birkhoff and D. Lewis, 1946)
$\chi(G,a) \neq 0$ *for* G *planar and* $a \in \mathbb{R}, a \geq 5$*.*

Note that this is much stronger than the 5-color theorem.

Theorem 6 (K. Appel and W. Haken, 1977)
Every planar graph is 4-colorable.
$\chi(G,4) \neq 0$ *for* G *planar.*

Problem 7
Find an analytic proof of the 4-color theorem.

Conjecture 8 (G. Birkhoff and D. Lewis, 1946)
For G *planar, there are no real roots of* $\chi(G,a)$ *for* $4 \leq a \leq 5$*.*

Real roots of $\chi(G, X)$

We note that $\chi(G, 0) = 0$ always, and $\chi(G, 1) = 0$ any graph with at least one edge.

Theorem 9 (D. Woodall, 1977)
Let G be any graph.

- *There are no negative real roots of $\chi(G, X)$.*
- *There are no real roots of $\chi(G, X)$ in the open interval $(0, 1)$.*

Theorem 10 (B. Jackson, 1993)

- *There are no real roots of $\chi(G, X)$ in the semi-open interval $(1, \frac{32}{27}]$.*
- *For any $\epsilon > 0$ there is a graph G_ϵ such that $\chi(G_\epsilon, X)$ has a root in $(\frac{32}{27}, \frac{32}{27} + \epsilon)$.*

Theorem 11 (S. Thomassen, 1997)
For any real numbers a_1, a_2 with $\frac{32}{27} \leq a_1 < a_2$ there exists a graph G such that $\chi(G, X) = 0$ for some $a \in (a_1, a_2)$.

Other counting interpretations: Acyclic orientations

An **orientation** of a graph G is a function which for every edge $e = (a, b)$ selects a source value $s(e) \in \{a, b\}$

An orientation is **acyclic**, of there are no oriented cycles.

Theorem 12 (R.P. Stanley, 1993)
The number of acyclic orientations of a graph G is given by the absolute value $\mid \chi(G, -1) \mid$.

Subgraph expansions

Let G be a graph with $k(G)$ connected components.
Let $S \subset E(G)$ and denote by $\langle S \rangle$ the subgraph generated by S in G.

- The **rank** $r(G)$ is defined as $r(G) = \mid V(G) \mid - k(G)$.
- The **corank** $s(G)$ is defined as $s(G) = \mid E(G) \mid - \mid V(G) \mid + k(G)$.
- The **rank polynomial** of a graph is defined by

$$R(G; X, Y) = \sum_{S \subseteq E(G)} X^{r(\langle S \rangle)} Y^{s(\langle S \rangle)}$$

Theorem 13 (H. Whitney, 1932)

(i) $\chi(G, X) = \sum_{S \subseteq E(G)} (-1)^{|S|} X^{|V(G)| - r(\langle S \rangle)}$

(ii) $\chi(G, X) = X^{|V|} R(G, -X^{-1}, -1)$

Definability of the chromatic polynomial.

- From Whitney's Theorem we get:

  $$\chi(G, X) = \sum_{S \subseteq E(G)} (-1)^{|S|} X^{|V(G)| - r(\langle S \rangle)}$$

  is definable in MSOLEVAL$(\tau_{hgraph}$.

- We shall see later that $\chi(G, X)$ is not definable in MSOLEVAL$(\tau_{graph}$

The complexity of the chromatic polynomial, I

Let us look at the chromatic polynomial $\chi(G, X)$.

- $\chi(G, X)$ has integer coefficients,
  and for $X \geq 0$ non-negative values,
  hence evaluating it at $X = a, a \in \mathbb{N}$ is in $\sharp\mathbf{P}$.

- For $a = 0, 1, 2$ evaluating $\chi(G, X)$ is in $\mathbf{P}$.

- For integer $a \geq 3$ evaluating $\chi(G, X)$ is $\sharp\mathbf{P}$-complete.

- What about evaluating $\chi(G, X)$ for $X = b$ with
  $b \in \mathbb{Z}$, $b \leq 0$?
  $b \in \mathbb{R}$ or $b \in \mathbb{C}$?

Given evaluations of $\chi(G, X)$ at $\mid V(G) \mid +1$ many points, we can compute the coefficients of $\chi(G, X)$ efficiently.

The complexity of the chromatic polynomial, II

Theorem 14
(F. Jaeger, D. Vertigan and D. Welsh, 1990)

For any two points $a, b \in \mathbb{C}$ *different from* $0, 1, 2$,
there is a **polynomial time algebraic reduction**
from the evaluation of $\chi(G, a)$ *to the evaluation of* $\chi(G, b)$.

Hence they are all **equally difficult**.

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There are a few problems with the exact formualtion of the theorem:

- What is the computational model behind **polynomial time algebraic reductions**?
- What is the computational model behind **equally difficult**.
- The hardness result is obtained by a reduction to $\sharp\mathbf{P}$-complete problem, but most instances are not in $\sharp\mathbf{P}$.

End of digression on typical theorems about the chromatic polynomial

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Example 6 *The characteristic polynomial*

- Let $V = [n]$ and let A_G be the (symmetric) adjacency matrix of G with $(A)_{j,i} = (A)_{i,j} = 1$ iff there is an edge between vertex i and vertex j.

- We denote by $P(G, X)$ the polynomial

  $$\det(X \cdot \mathbf{1} - A)$$

  $P(G, X)$ is a graph invariant and a polynomial in X,
  called the **characteristic polynomial of** G.

- The set of roots of $P(G, X)$ (with multiplicities) are the eigenvalues of A_G, and are called the **spectrum of the graph** G.

The characteristic polynomial and the spectrum of a graph
was first studied in the 1950ties

T.H. Wei 1952, L.M. Lihtenbaum 1956,
L. Collatz and U. Sinogowitz 1957,
H. Sachs 1964, H.J. Hoffman 1969

The characteristic polynomial: Literature

The characteristic polynomial and spectra of graphs have a very rich literature with important applications in chemistry under the name **Hückel theory**.

- N. Biggs, Algebraic Graph Theory,
  Cambridge University Press, 1994 (2nd edition)

- D.M. Cvetković, M. Doob and H. Sachs
  Spectra of Graphs
  Johann Ambrosius Barth, 1995 (3rd edition)

- D.M. Cvetković, P. Rowlinson and S. Simić
  Eigenspaces of Graphs
  Encyclopedia of Mathematics, vol. 66
  Cambridge University Press, 1997

- N. Trinajstić
  Chemical Graph Theory
  CRC Press, 1992 (2nd edition)

Digression 2:

Typical theorems about the characteristic polynomial

Coefficients of $P(G, X)$

We write

$$P(G, X) = \sum_{i=0}^{|V(G)|} c_i(G) \cdot X^{n-i}$$

Proposition 15

(i) $c_0 = 1$

(ii) $c_1 = 0$

(iii) $-c_2 = \mid E(G) \mid$ *is the number of edges of* G*.*

(iv) $-c_3$ *is twice the number of triangles of* G*.*

Eigenvalues of G, I

As in linear algebra, the zeros of $P(G, X)$ are called
eigenvalues of the matrix A_G, or **eigenvalues of the graph** G,

Proposition 16

(i) *All the eigenvalues of* G *are real.*

(ii) *If* G *is connected, the largest eigenvalue of* G *has multiplicity* 1.

(iii) *If* G *is connected and of diameter at least* d, *the* G *has at least* $d + 1$ *distinct zeros.*

(iv) *The complete graph is the only connected graph with exactly two distinct eigenvalues,* $P(K_n, X) = (X + 1)^{n-1}(X - n + 1)$.

(v) *Let* $\Lambda(G)$ *be the largest eigenvalue of* G.
G *is bipartite iff* $-\Lambda(G)$ *is also an eigenvalue of* G.

Eigenvalues of G, II

Proposition 17
Let G be a regular graph of degree r. Then

(i) *r is an eigenvalue of G*

(ii) *If G is connected, then the multiplicity of r is 1.*

(iii) *For any eigenvalue λ of G we have $\mid \lambda \mid \leq r$.*

(iv) *The multiplicity of the eigenvalue r is the number of connected components of G.*

Eigenvalues of G, III

$\lambda(G)$ denotes the smallest eigenvalue of G.
$\lambda_2(G)$ denotes the second largest eigenvalue of G.
$\Lambda(G)$ denotes the largest eigenvalue of G.

Proposition 18

(i) *If H is an induced subgraph of G, then $\lambda(H) \leq \lambda(G)$.*

(ii) *If H is an induced subgraph of G, then $\Lambda(H) \leq \Lambda(G)$.*
If H is a proper induced subgraph, then $\Lambda(H) < \Lambda(G)$.

(iii) *For no graph G is $\lambda(G) \in (-1, 0)$.*

(iv) *Let G have at least two vertices.*
$\lambda(G) = -1$ iff G is a complete graph.

(v) *For no graph G is $\lambda(G) \in (-\sqrt{2}, -1)$.*

(vi) *(J. Smith, 1970) $\lambda_2(G) \leq 0$ iff G is a complete multipartite graph.*

End of digression on typical theorems about the characteristic polynomial

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Example 7 *The acyclic or matching defect polynomial, I*

We denote by $m_k(G)$ the number of k-matchings of a graph G, with $m_0(G) = 1$ by convention.

- The polynomial

  $$m(G, X) = \sum_{k}^{\frac{n}{2}} (-1)^k m_k(G) X^{n-2k}$$

  is called the **acyclic polynomial of G** and also the **reference polynomial** or **matching defect polynomial**.

The acyclic or matching defect polynomial, II

The acyclic polynomial has important applications in Chemistry (Hückel theory again) and and Molecular Physics of Ferromagnetisms. It was first studied in the 1970 (Heilman and Lieb, Kunz)

- L. Lovász and M.D. Plummer
  Matching Theory
  Annals of Discrete mathematics, vol. 29
  North-Holland 1986

- N. Trinajstić,
  Chemical Graph Theory
  CRC, 1992 (2nd edition)

- P.J. Garratt
  Aromaticity
  John Wiley and Sons, 19xx

Example 8 *The matching (generating) polynomial*

- The polynomial

$$g(G,X) = \sum_{k}^{n} m_k(G) X^k$$

is called the **matching polynomial of** G or the **matching generating polynomial of** G.

- It is easy to verify the identity

$$m(G,X) = X^n g(G, (-X^{-2}))$$

Example 9 *Multi-variate graph polynomials*

Inspired by H. Whitney's work (1932) W.T. Tutte (1947, 1954) investigated generalizations of the chromatic polynomial to a polynomial in two variables, which he called the **dichromatic polynomial**, but now is called the **Tutte polynomial**, $T(G, X, Y)$.

The Tutte polynomial and its many generalizations became prominent, due to its many combinatorial interpretations in fields outside graph theory:

- Knot theory (via the Jones polynomial and its relatives)
- Statistical mechanics
- Quantum theory and quantum computing
- Chemistry

Example 10 *The Tutte polynomial*

Let $G = (V, E)$ be a graph,
and for $A \subseteq E$, let $G_A = (V, A)$ be a spanning subgraph.

The rank $r(G; A)$ is defined as $\mid V(G) \mid - k(G_A)$.

The **Tutte polynomial of** G is defined as

$$T(G; X, Y) = \sum_{A \subseteq E} (X - 1)^{r(G;E) - r(G;A)} \cdot (Y - 1)^{|A| - r(G;A)}$$

This looks confusing and innocent at the same time.

The fascination with the Tutte polynomial

The Tutte polynomial is like
a **magician's hat** with
rabbits, birds and other surprises coming out.

Easy manipulations produce various combinatorial counting functions. We have, at first glance surprisingly, the following

- $T(G, 1, 1)$ counts the number of spanning trees of G.
- $T(G, 2, 1)$ counts the number of forests of G.
- $T(G, 2, 0)$ counts the number of acyclic orientations of G.
- The chromatic polynomial is given by
$$\chi(G, X) = (-1)^{r(G;E)} X^{k(G)} T(G; 1 - X, 0)$$
- The reliability polynomial and the flow polynomial can also be derived with similar formulas.

Definition 11 *Complete graph invariants*

A graph invariant f is **graph-complete** if for any two graphs G_1, G_2 with $f(G_1) = f(G_2)$ we have also $G_1 \simeq G_2$.

The following is a graph-complete graph invariant.

- Let $X_{i,j}$ and Y be indeterminates.
  For a graph $\langle V, E \rangle$ with $V = [n]$ we put

$$Compl(G, Y, \bar{X}) = Y^{|V|} \cdot \left(\sum_{\sigma \in \mathfrak{S}_n} \prod_{(i,j) \in E} X_{\sigma(i),\sigma(j)} \right)$$

  Here $\mathfrak{S}_n$ is the permutation group of $[n]$.

Challenge: Find a polynomial in a constant finite number of indeterminates which is a graph-complete graph invariant.

An “unnatural” graph-complete invariant

Let $g : \mathcal{G} \rightarrow \mathbb{N}$ be a Gödel numbering for labeled graphs of the form $G = \langle [n], E, <_{nat} \rangle$.

We define a graph polynomial using g:

$$\Gamma(G, X) = \sum_{H \simeq G} X^{g(H)}$$

Clearly this is a graph invariant.

But it is “obviously unnatural” !

Can we make precise

what a **natural** graph polynomial should be?

Comparing graph invariants

In the literature we often find statements or questions of the form

- The Tutte polynomial is generalization of the chromatic polynomial.
- The Tutte polynomial does not determine the matching polynomial.
- Is there a natural most general graph polynomial?

We attempt to make this precise

Definition 12 *Induced graph invariants*

Let $\mathcal{H} \subseteq \mathcal{G}$ be a class of graphs closed under isomorphisms.
Let F be a set of graph invariants in a ring $\mathcal{R}$,
and let g be one more graph invariant.

We say that F **induces** g **on** $\mathcal{H}$,
or g **is a consequence of** F,
if for any two graphs $G_1, G_2 \in \mathcal{H}$ such that $f(G_1) = f(G_2)$ for all $f \in F$
we also have $g(G_1) = g(G_2)$.

We denote by $Ind_{\mathcal{R}}^{\mathcal{H}}(F)$ the set of graph invariants in $\mathcal{R}$ induced by F on $\mathcal{H}$.

We write also $F \models_{\mathcal{R}}^{\mathcal{H}} g$ for $g \in Ind_{\mathcal{R}}^{\mathcal{H}}(F)$.

How do we see if $F \models_{\mathcal{R}}^{\mathcal{H}} g$?

Example 13

Algebraically derived invariants

Let f, g be two graph invariants in $\mathcal{R}$.
Then the following are derived invariants of $F = \{f, g\}$:

- $f + g, f - g, f \times g$
- The formal derivative f'.
- Let $\phi : \mathcal{R}^2 \rightarrow \mathcal{R}$ be a function.
  Then $\phi(f, g)$ is induced by F.

Examples 14

Invariants induced by the characteristic polynomial

The characteristic polynomial $P(G, X)$ induces

- The number of vertices $\mid V \mid$.
- The number of edges $\mid E \mid$.
- The number of triangles of G.

We also have $P(K_{1,4}, X) = P(C_4 \sqcup E_1, X)$
but $K_{1,4}$ has no 2-matchings, whereas C_4 does.
Hence the $P(G, X)$ does not induce the number of connected components $k(G)$ nor $m(G, X)$.

Example 15

Invariants induced by the acyclic polynomial.

The acyclic polynomial $m(G, X)$ induces

- The number of vertices $\mid V \mid$.
- The number of edges $\mid E \mid$.
- The number of perfect matchings.
- the matching generating polynomial.

On the otherside $m(E_n, X) = 1$ for all $n \in \mathbb{N}$,
whereas $P(E_n, X) = X^n$.
Hence the $m(G, X)$ does not induce the characteristic polynomial $P(G, X)$.

Example 16

Invariants induced by the chromatic polynomial

The following are induced by $\chi(G,X) = \sum_{i=1}^{n} (-1)^{n-i} h_i X^i$:

- The cardinality of $V(G) = n$ is the degree of $\chi(G,X)$.
- The cardinality of $E(G) = m = h_{n-1}$.
- The chromatic number $\chi(G)$ is
  the smallest integer a such that $\chi(G,a) > 0$.
- The number of connected components $k(G)$
  is the multiplicity of zeros $X = 0$.
- The number of blocks $b(G)$ is the multiplicity of zeros $X = 1$.
- The girth $g = g(G)$ is given by the fact that
  for $0 \leq i \leq g-2$ we have $h_{n-i} = \binom{E(G)}{i}$.

Example 17

The acyclic polynomial and the characteristic polynomial.

Theorem 18 (I. Gutman, 1977)

$P(G, X) = m(G, X)$ iff G is a forest.

For $\mathcal{H} = \mathcal{F}$ the forests we have

$$P(G, X) = m(G, X)$$

i.e., the acyclic (matching defect) polynomial and the characteristic polynomial coincide,
and we have

$$P(G, X) \models^{\mathcal{F}} m(G, X) \text{ and } m(G, X) \models^{\mathcal{F}} P(G, X).$$

and

$$P(G, X) \models^{\mathcal{F}} g(G, X) \text{ and } g(G, X) \models^{\mathcal{F}} P(G, X).$$

In general, none induces the other.

I. Gutman, The acyclic polynomial of a graph, Publ. Inst. Math. (Beograd) (N.S.) 22 (36) (1977), pp. 63-69.

Example 19

The acyclic polynomial and the chromatic polynomial.

Definition 20
The complement graph of the simple graph $G = (V, E)$
is the graph $\bar{G} = (V, V^2 - D(V) - E)$.

For a graph polynomial $g = g(G, \bar{X})$ *the* **adjoint polynomial** $\hat{g}(G, \bar{X})$ *of* g *is defined by* $\hat{g}(G, \bar{X}) = g(\bar{G}, \bar{X})$.

Theorem 21 (E.J. Farrell and E.G. Whitehead Jr. 1992)
For $\mathcal{H} = \mathcal{TF}$, *the triangle free graphs, we have*

$$\hat{\chi}(G, X) \models^{\mathcal{TF}} m(G, X) \text{ and } m(G, X) \models^{\mathcal{TF}} \hat{\chi}(G, X).$$

i.e., the acyclic (matching defect) polynomial
and the adjoint chromatic polynomial mutually induce each other.

Note that $\chi(P_4) = \chi(K_{1,3})$, $P_4 \simeq \bar{P}_4$, *but* $m(P_4) \neq m(K_{1,3})$. *On the other hand,* $m(E_n) = 1$ *for each* $n \in \mathbb{N}$, *and* $\chi(E_n) = X^n$.
Hence, in general, none induces the other.

Example 22

The chromatic polynomial and Tutte polynomial

(i) The chromatic polynomial $\chi(G, X)$ is not induced by the Tutte polynomial $T(G, X, Y)$.

(ii) On connected graphs $\mathcal{C}$ we have $T(G, X, Y) \models^{\mathcal{C}} \chi(G, X)$ for

(iii) Tutte polynomial $T(G, X, Y)$ is not induced by the the chromatic polynomial $\chi(G, X)$.

Proof:

(i) Let E_n be the graph with n vertices and no edges. We have $T(E_n, X, Y) = 1$ but $\chi(E_n, X) = X^n$.

(ii) (After W.T. Tutte, 1954) $\chi(G, X) = (-1)^{|V|-k(G)} X^{k(G)} T(G, 1-X, 0)$.

(iii) (After M. Noy, 2003) Let W_n be the wheel with n spokes. It is known that $T(G, X, Y) = T(W_n, X, Y)$ implies that $G \simeq W_n$ for all n.

But there is a $G \not\simeq W_5$ with $\chi(G, X, Y) = \chi(W_5, X, Y)$.

Example 23

The Tutte polynomial and the matching polynomials

- The matching polynomial is not induced by the Tutte polynomial, even on connected planar graphs.
- The Tutte polynomial is not induced by the matching polynomial, even on connected planar graphs.

Proof:

(i) For trees with n vertices t_n we have $T(t_n, X, Y) = X^{n-1}$. But it is easy to see that $K_{1,n-1}$ and P_n are both trees with n vertices and their matching polynomials differ, as $K_{1,n-1}$ has no 2-matching but P_n has for $n \geq 3$.

(ii) On the other hand $C_3 \sqcup_e C_5$ and $C_4 \sqcup_e C_4$ have the same matching polynomials (check by hand) but have different Tutte polynomials, as the Tutte polynomials counts cliques of given size.

What do we learn?
What do we ask?

- Polynomial graph invariants are still a mystery.
- Can we analyze the consequence relation for polynomial invariants?
- Can we identify “good invariants”?
- What are appropriate complexity classes for graph invariants?

Outline of the course

LECTURE 00: Second Order Logic (SOL) and its fragments (Background, not lectured) LOGICS (14 slides)

LECTURE 01: Classical graph properties and graph parameters and their definability in SOL (4 hours) G-PARAMETERS, (60 slides)

LECTURE 02: One, two, many graph polynomials (4 hours) LANDSCAPE, (ca. 50 slides)

LECTURE 03: The characteristic and the matching polynomial (4 hours MATCHING, (54 slides)

LECTURE XX: Graph polynomials in Physics and Chemistry (2 hours) CHEMISTRY, (38 slides)