

Lecture 1: Prelude

In this course we look at (labeled) graphs and other relational structures.

- Graph properties are classes of graphs closed under graph isomorphism.
- Graph parameters are functions of graphs invariant under graph isomorphism with values in some domain, usually a ring or semi-ring such as the natural numbers \mathbb{N} or the integers \mathbb{Z} or the reals \mathbb{R} , or a polynomial ring in several indeterminates.

Second Order Logic (SOL)

- Second Order Logic is the **natural language** to talk about **graph properties**.
We shall show this **informally** and only after that define the **syntax** and **semantic** of SOL.
- We shall see we can also use SOL to define **graph parameters**.

Second Order Logic SOL and some of its fragments.

Atomic formulas for **graphs** are $E(u, v)$ and $u = v$ for individual variables u, v , and $R(u_1, \dots, u_m)$ for m -ary relation variables R .

- **First Order Logic FOL:**

Closed under boolean operations and quantification over **individual variables**. **No relation variables**.

- **Second order Logic SOL:**

Closed under boolean operations and quantification over individual and **relation variables of arbitrary but fixed arity**.

- **Monadic Second order Logic MSOL:**

Closed under boolean operations and quantification over individual and **unary relation variables**.

Concrete graphs (in \mathbb{R}^3)

A **concrete** graph G is given by

- a finite set of **points** V in \mathbb{R}^3 , and
- a finite set E of **ropes** linking two points v_1, v_2 .

The **ropes** are **continuous curves** which **do not intersect**.

Without loss of generality we can take the points also in \mathbb{R}^m for $m \geq 3$.

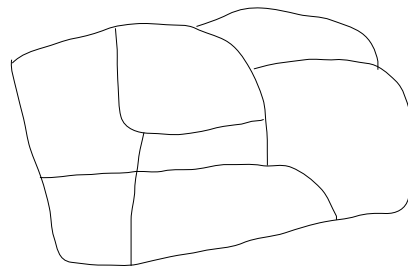
The **ropes** are called **arcs**.

Plane graphs

A **plane** graph G is given by

- a finite set of **points** V in \mathbb{R}^2 , and
- finite set E of **arcs** linking two points v_1, v_2 .

The arcs are **continuous curves** which do not intersect.



All intersection points in the drawing are points of the graph!

Abstract graphs

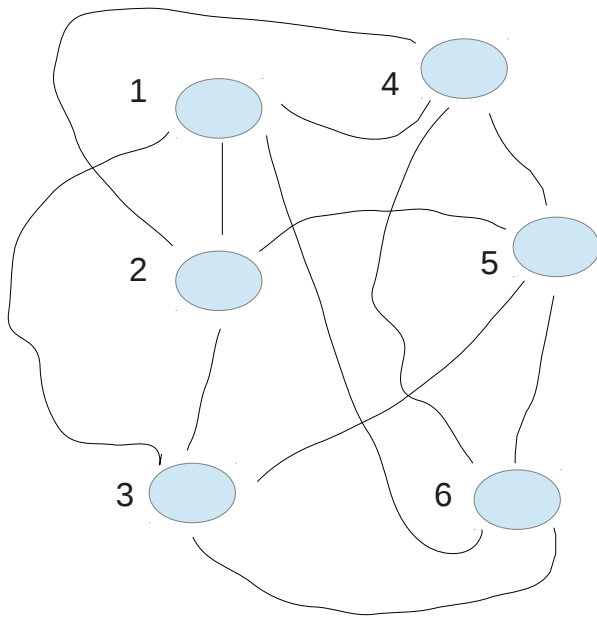
An **abstract** graph $G = (V(G), E(G))$ is given by

- a finite set of **vertices** $V = V(G)$, and
- a finite set $E = E(G)$ of **edges** linking two vertices v_1, v_2 .

Here $E \subseteq V^{(2)}$ where $V^{(2)}$ denotes the set of **unordered pairs** of elements of V .

$$V = \{1, \dots, 6\}$$

$$E = \left\{ \begin{array}{l} \{(1, 2), (2, 3), (3, 1)\} \cup \\ \{(4, 5), (5, 6), (6, 4)\} \cup \\ \{(1, 6), (6, 3), (3, 5), (5, 2), (2, 4), (4, 1)\} \end{array} \right.$$



Graph isomorphism and subgraphs

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there is a function $f : V_1 \rightarrow V_2$ such that

- f is bijective (one-one and onto), and
- $(u, v) \in E_1$ iff $(f(u), f(v)) \in E_2$.

$G_1 = (V_1, E_1)$ is a **subgraph** of $G_2 = (V_2, E_2)$ if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$.

$G_1 = (V_1, E_1)$ is an **induced subgraph** of $G_2 = (V_2, E_2)$ if $V_1 \subseteq V_2$ and for all $(u, v) \in V_1^{(2)} \cap E_2$ we also have $(u, v) \in E_1$.

$G_1 = (V_1, E_1)$ is a **spanning subgraph** of $G_2 = (V_2, E_2)$ if $E_1 \subseteq E_2$ and for all $(u, v) \in E_1$ both $u, v \in V_1$.

Two isomorphic graphs

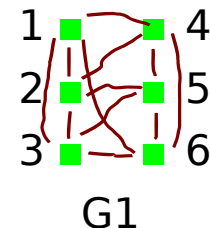
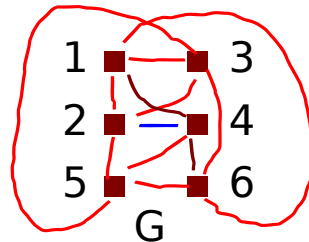
$$V_1 = V_2 = \{1, \dots, 6\}$$

$$E_1 = \left\{ \begin{array}{l} \{(1, 2), (2, 3), (3, 1), (4, 5), (5, 6), (6, 4)\} \cup \\ \{(1, 6), (6, 3), (3, 5), (5, 2), (2, 4), (4, 1)\} \end{array} \right.$$

$$E_2 = \left\{ \begin{array}{l} \{(1, 4), (4, 3), (3, 1), (5, 2), (2, 6), (6, 5)\} \cup \\ \{(1, 6), (6, 3), (3, 2), (2, 4), (4, 5), (5, 1)\} \end{array} \right.$$

G_1 and G_2 are isomorphic with

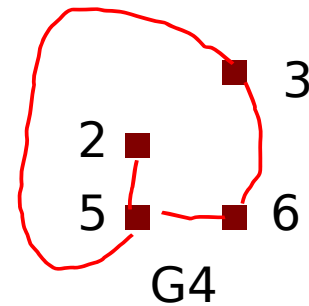
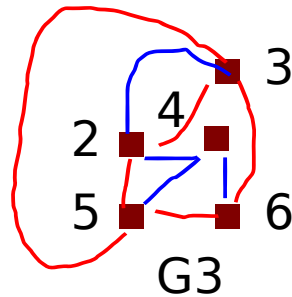
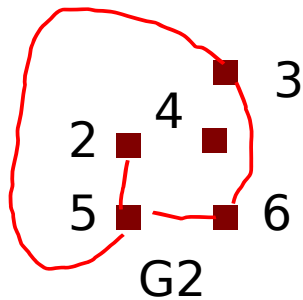
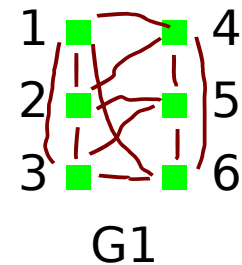
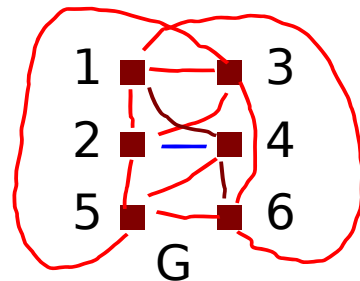
$$f(1) = 1, f(2) = 4, f(3) = 3, f(4) = 5, f(5) = 2, f(6) = 6.$$



G_1 is isomorphic to G .

G_2 is a subgraph of G , but not an induced subgraph.

G_3 is an induced subgraph and G_4 is a spanning subgraph of G .



Some graph properties: Regularity

A graph G is (give definition in SOL):

- of **degree bounded** by $d \in \mathbb{IN}$.
Every vertex has **at most** d neighbors.
- k -regular ($k \in \mathbb{IN}$)
Every vertex has **exactly** k neighbors.
- regular
Every vertex has exactly the **same number** of neighbors.
- Regular and degree bounded by d .

Definable in First Order Logic FOL

- The vertices v_0, v_1, \dots, v_n are all different:

$$\text{Diff}(v_0, v_1, \dots, v_n) : \left(\bigwedge_{\substack{i,j \leq n \\ i=0, j=1, i < j}} v_i \neq v_j \right)$$

- A vertex v_0 has degree at most d :

$$\text{Deg}_{\leq d}(v_0) : \forall v_1, \dots, v_d, v_{d+1} \left(\bigwedge_{i=0}^{d+1} E(v_0, v_i) \rightarrow \bigvee_{\substack{i=d+1, j=d+1 \\ i=0, j=0, i \neq j}} v_i = v_j \right)$$

- A vertex v_0 has degree at least d :

$$\text{Deg}_{\geq d}(v_0) : \exists v_1, \dots, v_d \left(\text{Diff}(v_1, \dots, v_d) \wedge \bigwedge_{i=1}^d E(v_0, v_i) \right)$$

Regularity definable in

The following graph properties are definable in FOL (use previous slide):

- k -regular;
- regular and of bounded degree d ;

The following are **not** definable in FOL (nor in Monadic Second order Logic MSOL):

- regular;
- each vertex has even degree.

To show non-definability in FOL we need the machinery of **Ehrenfeucht-Fraïssé Games** or **Connection matrices**.

Regularity definable in

The following are definable in SOL:

- Two sets $A, B \subseteq V$ have the same size:

$$\text{EQS}(A, B) : \exists R (\text{Funct}(R, A, B) \wedge \text{Inj}(R) \wedge \text{Surj}(R))$$

where $\text{Funct}(R, A, B)$, $\text{Inj}(R)$, $\text{Surj}(R)$ are FOL-formulas saying that R is a function from A to B which is one-one (injective) and onto (surjective).

- A vertex v has even degree:

The set of neighbors of v can be partitioned into two sets of equal size

$$\text{EDeg}(v_0) : \exists A, B (\text{Part}(N_v, A, B) \wedge \text{EQS}(A, B))$$

- Two vertices u, v have the same degree:

The set of neighbors N_u, N_v of u and v have the same size.

$$\text{SDeg}(u, v) : \text{EQS}(N_u, N_v)$$

Some graph properties: Closure properties of graph classes.

A graph property is called

- **hereditary** if it is closed under **induced subgraphs**.
- **monotone** if it is closed under **subgraphs**, not necessarily induced.
- **monotone decreasing** if it is closed under **deletion of edges**, but not necessarily of vertices.
- **monotone increasing** if it is closed under **addition of edges**, but not necessarily of vertices.
- **additive** if it is closed under **disjoint unions**.

Note that **monotone** implies **hereditary** and **monotone decreasing**.

Examples for the closure properties

- Regular graphs are only additive.
- Graphs of bounded degree d are monotone and additive.
- Cliques (complete graphs) are hereditary but not monotone.
- Connectivity is only monotone increasing.
- **Exercise:** Check the above closure properties of graph properties for your favorite graph properties.
- **Exercise:** Check the above closure properties of all the graph properties discussed in the [sequel of this course](#).

Forbidden (induced) subgraphs

Let $\mathcal{H} = \{H_i : i \in I\}$ be a family of graphs.

- We denote by $\text{Forb}_{sub}(\mathcal{H})$ ($\text{Forb}_{ind}(\mathcal{H})$) the class of graphs G which have no (induced) subgraph isomorphic to some graph $H \in \mathcal{H}$.
- $\text{Forb}_{sub}(\mathcal{H})$ is monotone and $\text{Forb}_{ind}(\mathcal{H})$ is hereditary.

Theorem: (**Exercise**)

Let \mathcal{P} be a monotone (hereditary) graph property. Then there exists a family $\mathcal{H} = \{H_i : i \in I\}$ of finite graphs such that $\mathcal{P} = \text{Forb}_{sub}(\mathcal{H})$ (respectively $\mathcal{P} = \text{Forb}_{ind}(\mathcal{H})$).

Proposition: Let $\mathcal{H} = \{H_i : i \in I\}$ be a family of graphs with I finite. Then **both** $\text{Forb}_{sub}(\mathcal{H})$ and $\text{Forb}_{ind}(\mathcal{H})$ are **definable in FOL**.

Homework 1

Characterize the following graph properties using $\text{Forb}_{sub}(\mathcal{H})$ or $\text{Forb}_{ind}(\mathcal{H})$, and determine their definability in FOL and SOL.

- Forests
- Cliques
- Find other examples! You may consult:

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@BOOK(bk:BrandstaedtLeSpinrad,  
AUTHOR      = {A. Brandst\"adt and V.B. Le and J. Spinrad},  
TITLE       = {Graph Classes: A survey},  
PUBLISHER   = {{SIAM} },  
SERIES      = {{SIAM} Monographs on Discrete Mathematics and Applications},  
YEAR       = {1999})
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Some graph properties: Colorability

Let \mathcal{P} be a graph property. A graph G is (*give definition in SOL, MSOL*):

- **3-colorable:**

The vertices of G can be partitioned into three disjoint sets $C_i : i = 1, 2, 3$ such that the induced graphs $G[C_i]$ consist only of isolated points.

This can be expressed in MSOL.

- **k - \mathcal{P} -colorable ($k \in \mathbb{IN}$):**

The vertices of G can be partitioned into k disjoint sets $C_i : i = 1, \dots, k$ such that the induced graphs $G[C_i]$ are in \mathcal{P} .

If \mathcal{P} is definable in SOL (MSOL), this is also definable in SOL (MSOL).

- **\mathcal{P} -colorable:**

The vertices of G can be partitioned into disjoint sets $C_i : i \in I \subset \mathbb{IN}$ such that the induced graphs $G[C_i]$ are in \mathcal{P} .

This is definable in SOL provided \mathcal{P} is. It is not MSOL-definable.

k -colorable graphs

A subset V_1 of a graph $G = (V, E)$ is **independent** if it induces a graph of isolated points (without neighbors nor loops).

A graph is **k -colorable** if its vertices can be partitioned into k independent sets.

$$\begin{aligned} & \text{Part}(X_1, X_2, X_3) : \\ & ((X_1 \cup X_2 \cup X_3 = V) \wedge ((X_1 \cap X_2) = (X_2 \cap X_3) = (X_3 \cap X_1) = \emptyset)) \end{aligned}$$

$$\begin{aligned} & \text{Ind}(X) : \\ & (\forall v_1 \in X)(\forall v_2 \in X) \neg E(v_1, v_2) \end{aligned}$$

With this 3-colorable can be expressed as

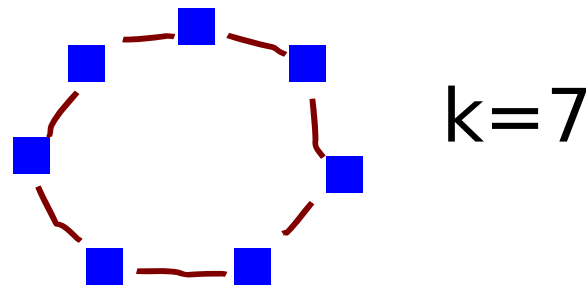
$$\exists C_1 \exists C_2 \exists C_3 (Part(C_1, C_2, C_3) \wedge Ind(C_1) \wedge Ind(C_2) \wedge Ind(C_3))$$

We have expressed 3-colorability by a formula in **Monadic Second Order Logic**.

Question: Can we express this in First Order Logic ?

Some graph properties: Chordality

A graph is a **simple cycle of length k** if it is of the form:



A graph is a simple cycle iff it is **connected** and **2-regular**.

A graph G is **chordal** or **triangulated** if there is no induced subgraph of G isomorphic to a simple cycle of length ≥ 4 .

Exercise: Find a MSOL-expression for chordality.

Some graph properties: Eulerian and Hamiltonian

A graph G is (*give definition in SOL*):

- **Eulerian:**

We can follow each edge exactly once, pass through all the edges, and return to the point of departure.

Theorem (Euler): A graph is Eulerian iff it is connected and each vertex has even degree.

- **Hamiltonian:**

We can follow the edges visiting each vertex exactly once, and return to the point of departure.

Eulerian graphs

A graph $G = (V, E)$ is **Eulerian** if we can follow each edge exactly once, pass through all the edges, and return to the point of departure.

Equivalently:

Can we order all the edges of E

$$e_1, e_2, e_3, \dots, e_m$$

and choose beginning and end of th edge $e_i = (u_i, v_i)$ such that for all i , $v_i = u_{i+1}$ and $v_m = u_1$.

$$\begin{aligned} &\exists R (\text{LinOrd}(R, E) \wedge \\ &(\forall u, v, u', v' \text{First}(R, u, v) \wedge \text{Last}(R, u', v') \rightarrow u = v') \wedge \\ &(\forall u, v, u', v' \text{Next}(R, u, v, u'v') \rightarrow v = u')) \end{aligned}$$

whith the obvious meaning of $\text{LinOrd}(R, E)$, $\text{First}(R, u, v)$ and $\text{Last}(u, v)$.

Alternatively, we can use **Euler's Theorem**.

As we shall see later, it **cannot** be expressed in MSOL.

Hamiltonian graphs

We note: A graph with n vertices is Hamiltonian if it contains a spanning subgraph which is a cycle of size n .

We define formulas:

$\text{Conn}(V_1, E_1)$: (V_1, E_1) is connected.

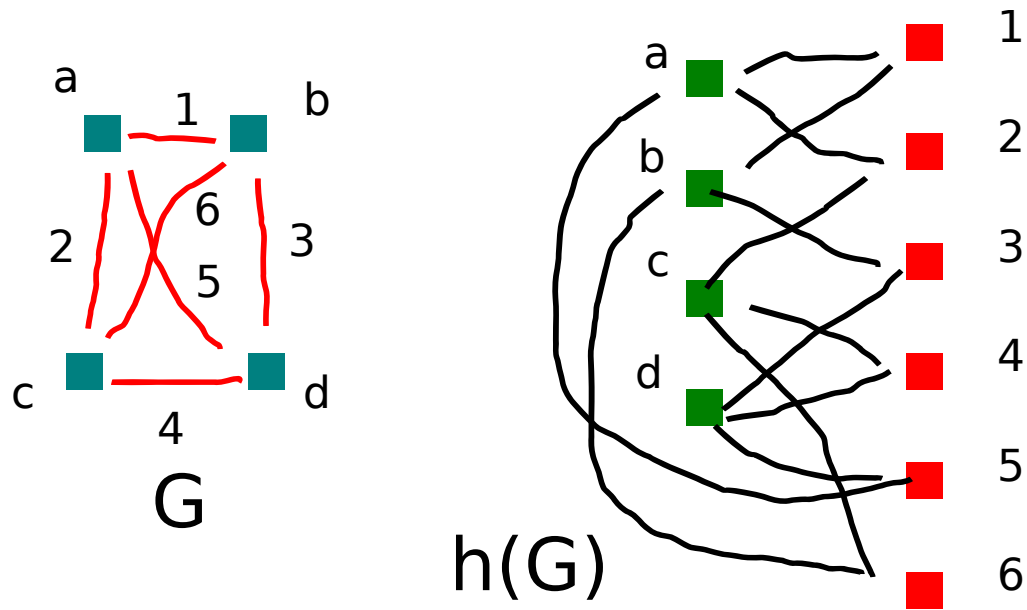
$\text{Cycle}(V_1, E_1)$: (V_1, E_1) is a cycle, i.e., regular of degree 2 and connected.

$\text{Ham}(V, E) : \exists V_1 \exists E_1 (\text{Cycle}(V_1, E_1) \wedge E_1 \subseteq E \wedge V_1 = V)$

A subtle point: Graphs vs hypergraphs, I

- **Graphs** are structures with universe V of vertices, and a **binary edge relation** E .
There can be at most one edge between two vertices.
- **Hypergraphs** have as their universe two disjoint sets V and E and an **incidence (hyperedge) relation** $R(u, v, e)$.
There can be many edges between two vertices.
- In both cases the relations are symmetric in the vertices.
- A Graph G can be viewed as hypergraph (h-graph) $h(G)$ where there is at most one edge (up to symmetry) between two vertices.
- There is a one-one correspondence between graph and h-graphs.

G and $h(G)$



A subtle point: Graphs vs hypergraphs, II

- FOL and SOL are **equally expressive** on graphs and h-graphs.
- MSOL is **more expressive** on h-graphs than on graphs.
Hamiltonicity is **not definable** in MSOL on graphs, but **is definable** on h-graphs.

We shall discuss this in detail in a **later lecture**.

How to prove definability in SOL, MSOL and FOL?

So far we have looked at properties of
abstract (directed) graphs and hypergraphs.

- Formulate the property using set theoretic language of finite sets over the set of vertices and edges and their incidence relation.
- Try to mimick this formulation in SOL.
- If you succeed, try to do it in MSOL or even FOL.

Test your fluency in SOL! (Homework)



Express the following properties in FOL, **if possible**.

- A graph G is a **cograph** if and only if there is no induced subgraph of G isomorphic to a P_4 .
- A G is **P_4 -sparse** if no set of 5 vertices induced more than one P_4 in G .
- **Triangle-free** graphs: There is no induced K_3 .
- Existence of prescribed (induced) subgraph H .
- **H -free** graphs: non-existence of prescribed (induced) subgraph H .
- Let \mathcal{P} be a graph property.
 \mathcal{P} -free graphs: non-existence of an induced subgraph $H \in \mathcal{P}$.

Topological properties of graphs (from Wikipedia)

[http://en.wikipedia.org/wiki/Genus_\(mathematics\)](http://en.wikipedia.org/wiki/Genus_(mathematics))

So far our graph properties were formulated in the language of graphs, involving as basic concepts only **vertices**, **edges** and their **incidence relations**.

Topological graph theory studies the **embedding of graphs** in **surfaces**, **spatial embeddings of graphs**, and graphs as **topological spaces**.

- A graph is **planar** if it is isomorphic to a plane graph.
- The **genus of a graph** is the minimal integer n such that the graph can be drawn without crossing itself on a sphere with n handles (i.e. an oriented surface of genus n).

Thus, a planar graph has genus 0, because it can be drawn on a sphere without self-crossing.



genus: 0, 1, 2, 3

Planar graphs, I

A graph is **planar** iff it is isomorphic to a plane graph.

This definition involves the **geometry** of th Euclidean plane.

How can we express planarity
without geometry ?

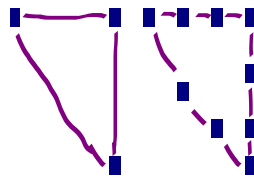


Kuratowski's Theorem

Kazimierz Kuratowski (1896-1980)

http://en.wikipedia.org/wiki/Kuratowski's_theorem

A **subdivision** of a graph G is a graph formed by subdividing its edges into paths of one or more edges.



K_3 and a subdivision of K_3

Theorem: A finite graph G is planar if and only if it does not contain a subgraph that is isomorphic to a subdivision of K_5 or $K_{3,3}$.

Planar graphs, II

Theorem: Planarity is definable in MSOL.

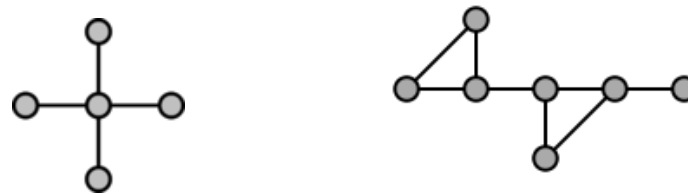
- We use Kuratowski's Theorem.
- For a fixed graph H , G is a subdivision of H , is definable in MSOL.
- For a graph property \mathcal{P} definable in MSOL, G has a subgraph $H \in \mathcal{P}$, is definable in MSOL.

Exercise: Prove the last two statements.

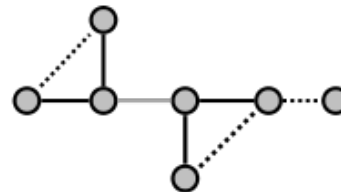
Graph minors, I

http://en.wikipedia.org/wiki/Graph_minor

An undirected graph H is called a **minor** of the graph G if H can be formed from G by **deleting edges** and **vertices** and by **contracting edges**.



H is a minor of G .



First construct a subgraph of G by **deleting** the dashed edges (and the resulting isolated vertex), and then **contract** the thin edge (**merging** the two vertices it connects).

Graph minors, II

Proposition: For fixed H the statement H is a minor of G is definable in MSOL.

- An edge contraction is an operation which removes an edge from a graph while simultaneously merging the two vertices it used to connect.
- An undirected graph H is a minor of another undirected graph G if a graph isomorphic to H can be obtained from G by contracting some edges, deleting some edges, and deleting some isolated vertices.
- The order in which a sequence of such contractions and deletions is performed on G does not affect the resulting graph H .
- Let $(V)H = \{v_1, \dots, v_m\}$. We have to find $V_1, \dots, V_m \subseteq V(G)$ which we all contract to a vertex u_i corresponding to v_i such that V_i connects to V_j iff $(v_i, v_j) \in E(H)$.
- The vertices in $V(G) - \bigcup_i^m V_i$ are discarded.

Minor closed graph classes

- H is a **topological minor** of G if G has a subgraph which is isomorphic to a subdivision of H .
- A graph property \mathcal{P} is closed under (topological) minors, if whenever $G \in \mathcal{P}$ and H is a (topological) minor of G the also $H \in \mathcal{P}$.

Examples:

- Trees are not closed under minors, but forests are.
- Graphs of degree at most 2 are minor closed, but graphs of degree at most 3 are not.
- Planar graphs are both closed under minors and topological minors.

Forbidden minors, I

Let $\mathcal{H} = \{H_i : i \in I\}$ be a family of graphs.

- We denote by $\text{Forb}_{\min}(\mathcal{H})$ ($\text{Forb}_{\text{tmin}}(\mathcal{H})$) the class of graphs G which have no (topological) minors isomorphic to some graph $H \in \mathcal{H}$.
- $\text{Forb}_{\min}(\mathcal{H})$ is closed under topological minors, is monotone and hence, hereditary.

Theorem: (**Exercise**)

Let \mathcal{P} be a graph property closed under (topological) minors. Then there exists a family $\mathcal{H} = \{H_i : i \in I\}$ of finite graphs such that $\mathcal{P} = \text{Forb}_{\min}(\mathcal{H})$ (respectively $\mathcal{P} = \text{Forb}_{\text{tmin}}(\mathcal{H})$).

Proposition: Let $\mathcal{H} = \{H_i : i \in I\}$ be a family of graphs with I finite. Then both $\text{Forb}_{\min}(\mathcal{H})$ and $\text{Forb}_{\text{tmin}}(\mathcal{H})$ are definable in MSOL.

The Graph Minor Theorem

(aka Robertson-Seymour Theorem)

Here is one of the deepest theorems in structural graph theory:

Theorem:

Let \mathcal{P} be a graph property closed under minors.

Then $\mathcal{P} = \text{Forb}_{\min}(\mathcal{H})$ with \mathcal{H} **finite**.

Corollary: Every graph property \mathcal{P} property closed under minors

is **definable in MSOL**.

Wagner's Theorem and Hadwiger's Conjecture

Theorem: A graph G is planar iff K_5 and $K_{3,3}$ are not minors of G .

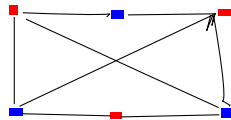
- This gives **another proof** that planarity is MSOL-definable.

Conjecture: If a graph G is not k -colorable then it has the complete graph K_{k+1} as a minor.

The conjecture was proven for $k \leq 6$.

The converse is not true.

There are bipartite graphs with a K_4 minor.



Logic and Complexity: Regular languages

Let $L \subseteq \Sigma^*$ be a *magenta language*, i.e., a set of words over the alphabet Σ .

We assume you are familiar with *automata theory*!

Theorem:(Kleene; Büchi, Elgot; Trakhtenbrot)

The following are equivalent:

- L is recognizable by a deterministic finite automaton.
- L is recognizable by a non-deterministic finite automaton.
- L is *regular*, i.e., describable by a *regular expression*
- The set of τ_{word} -structures \mathfrak{A}_w with $w \in L$ is definable in $\text{MSOL}(\tau_{word})$.

Complexity classes

We need to recall some complexity classes:

L: **Deterministic** logarithmic space.

NL: **Non-deterministic** logarithmic space.

P: **Deterministic** polynomial time.

NP: **Non-deterministic** polynomial time.

PH: The polynomial hierarchy.

#P: Counting predicates in **P** (Valiant's class)

PSpace: **Deterministic** polynomial space.

Complexity of SOL-properties

Fagin, Christen:

The **NP**-properties of classes of τ -structures are exactly the $\exists SOL$ -definable properties.

Meyer, Stockmeyer:

The **PH**-properties (in the *polynomial hierarchy*) of classes of τ -structures are exactly the SOL-definable properties.

Makowsky, Pnueli:

For every level Σ_n^P of **PH** there are *MSOL*-definable classes which are complete for it.

Separating Complexity Classes, I

We have

$$\mathbf{L} \subseteq \mathbf{NL} \subseteq \mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PH} \subseteq \#\mathbf{P} \subseteq \mathbf{PSpace}$$

- To show that **PH** **does not collapse** to **NP** we have to find a τ -sentence $\phi \text{SOL}(\tau)$ which is **not equivalent over finite structures** to an **existential** τ -sentence $\psi \text{SOL}(\tau)$.
- Every sentence $\phi \in \text{SOL}(\tau)$ is equivalent (over finite structures) to an existential sentence $\psi \in \text{SOL}(\tau)$ iff **NP = CoNP**.
Note we allow arbitrary arities of the quantified relation variables.
Over infinite structures this is known to be false (Rabin)
- If there is a $\phi \in \text{SOL}(\tau)$ which is **not** equivalent to an existential sentence, then **P \neq NP**.
And there should be such a sentence !
- To show that **PSpace** is different from **PH** it suffices to find a **PSpace**-complete graph property which is not SOL-definable.

HEX and Geography, I

- **The game HEX:**

Given a graph G and two vertices s, t .

Players I and II color alternately vertices in $V - \{s, t\}$ white and black respectively.

Player I tries to construct a white path from s to t and Player II tries to prevent this.

HEX: The class of graphs which allow a Winning Strategy for player I.

- **The game GEOGRAPHY:**

Given a **directed graph** G . Players I and II choose alternately new edges starting at the end point of the last chosen edge. The first who cannot find such an edge has lost.

GEO: The class of graphs which allow a Winning Strategy for I.

HEX and Geography, II

Theorem (Even, Tarjan): HEX is **PSPACE**-complete.

Theorem (Schaefer): GEO is **PSPACE**-complete.

Problem: Are they SOL-definable?

This would imply that **PSPACE = PH**, and the polynomial hierarchy **collapses** to some finite level!

Short versions: Fix $k \in \mathbb{N}$.

SHORT-HEX, SHORT-GEOGRAPHY asks whether Player I can win in k moves.

S-HEX and S-GEO are the class of (orderd) graphs where player I has a winning strategy.

S-HEX and S-GEO are FOL-definable for fixed k .
(and therefore solvable in **P**).

The role of order, I

Let $\tau_{=}$ be the one sorted vocabulary without any relation or constant symbols. We have only equality as atomic formulas.

Let $\tau_{<}$ be the one sorted vocabulary with one binary relation symbol $R_{<}$ which will be interpreted as a linear order.

- The class of structures of even cardinality EVEN is not definable in $\text{MSOL}(\tau_{=})$.

We shall prove this later.

- The class of structures of even cardinality EVEN is definable in $\text{MSOL}(\tau_{=})$ by a formula ϕ_{EVEN} .

The role of order, II: Constructing ϕ_{EVEN}

We use the order to define the binary relation $2NEXT$ and the unary relation Odd

- For a structure $\mathfrak{A} = \langle A, < \rangle$, let $(a, b) \in 2NEXT^{\mathfrak{A}}$ iff $a < b$ and there is exactly one element strictly between a and b .
- The first element is in $Odd^{\mathfrak{A}}$.
If $a \in Odd^{\mathfrak{A}}$ and $(a, b) \in 2NEXT^{\mathfrak{A}}$ then $b \in Odd^{\mathfrak{A}}$.
- Let ϕ_{EVEN} be the formula which says that the last element is not in Odd .
- Now the a structure $\langle A, < \rangle$ is in $EVEN$ iff its last element is not in $Odd^{\mathfrak{A}}$.

Q.E.D.

The role of order, III: Order invariance

In the previous example EVEN the MSOL($\tau_{<}$)-formula ϕ_{EVEN} is **order invariant** in the following sense:

Let $\mathfrak{A}_1, \mathfrak{A}_2$ be two structures with universe A and different order relations $<_1$ and $<_2$.

Then $\mathfrak{A}_1 \models \phi_{EVEN}$ iff $\mathfrak{A}_2 \models \phi_{EVEN}$.

We generalise this:

Let $\mathfrak{A}_1, \mathfrak{A}_2$ be two $\tau \cup \{R_{<}\}$ -structures with universe A and different order relations $\mathfrak{A}_1(R_{<}) = <_1$ and $\mathfrak{A}_2(R_{<}) = <_2$ but for all other symbols in $R \in \tau$ we have $\mathfrak{A}_1(R) = \mathfrak{A}_2(R)$.

A $\tau \cup \{R_{<}\}$ -formula in SOL is **order invariant** if for all structures $\mathfrak{A}_1, \mathfrak{A}_2$ as above we have

$$\mathfrak{A}_1 \models \phi \text{ iff } \mathfrak{A}_2 \models \phi$$

The fragment HornESOL(τ).

- A quantifier-free τ -formula is a **Horn clause** if it is a **disjunction** of atomic or negated atomic formulas where at most one is not negated.

$$\neg\alpha_1 \vee \neg\alpha_2 \vee \dots \vee \neg\alpha_n \vee \beta$$

where α_i, β are atomic.

- A quantifier-free τ -formula is a **Horn formula** if it is a **conjunction** of Horn clauses.
- A formula $\phi \in \text{SOL}(\tau)$ is in **HornESOL(τ)** if it is of the form

$$\exists U_{1,r_1}, U_{2,r_2}, \dots, U_{k,r_k} \forall v_1, \dots, v_m H(v_1, \dots, v_m, U_{1,r_1}, U_{2,r_2}, \dots, U_{k,r_k})$$

where H is a Horn formula and v_i are first order variables.

Some classes of graphs
order invariantly (o.i.) definable in $\text{HornESOL}(\tau_{\text{graph}})$

- Graphs of even cardinality, of even degree. **order is needed !**
- Bipartite graphs $G = (V_1, V_2, E)$ with $|V_1| = |V_2|$.
- Regular graphs, and regular graphs of even degree.
- Connected graphs.
- Eulerian graphs.

To be discussed on the blackboard.

The Immermann-Vardi-Graedel Theorem (IVG)

Let τ be a relational vocabulary with a binary relation for the ordering of the universe.

Theorem 1 (Immermann, Vardi, Graedel 1980-4)

Let \mathcal{C} be a set of finite τ -structures. The following are equivalent:

- $\mathcal{C} \in \mathbf{P}$;
- *there is a τ -formula $\phi \in \text{HornESOL}(\tau)$ such that $\mathfrak{A} \in \mathcal{C}$ iff $\mathfrak{A} \models \phi$.*

Here the presence of the ordering is crucial:

Without it the class of structures for the empty vocabulary of even cardinality is in \mathbf{P} , but not definable in HornESOL.

The Immermann-Vardi-Graedel Theorem (IVG):

Order invariant version

Let τ be a relational vocabulary and $\tau_1 = \tau \cup \{R_{<}\}$. with a binary relation *for the ordering of the universe*.

Theorem 2 (Graedel 1980-4, Dawar, Makowsky)

Let \mathcal{C} be a set of finite τ -structures. The following are equivalent:

- $\mathcal{C} \in \mathbf{P}$;
- *there is an order invariant τ_1 -formula $\phi \in \text{HornESOL}(\tau)$ such that for all τ -structures \mathfrak{A} and linear orderings $R^A \subset \mathfrak{A}(V)^2$ $\mathfrak{A} \in \mathcal{C}$ iff $\langle \mathfrak{A}, R^A \rangle \models \phi$.*

Conclusion: The logical equivalent to $P = NP$

Let τ be a relational vocabulary which contains a **binary relation for the ordering** of the universe.

The following are equivalent:

- **$P = NP$** in the classical framework.
- Every $ESOL(\tau)$ -formula is equivalent over **finite ordered τ -structures** to some $HornESOL(\tau)$ -formula.
- Every **o.i.** $ESOL(\tau)$ -formula is equivalent over **finite ordered τ -structures** to some **o.i.** $HornESOL(\tau)$ -formula.

Logics capturing complexity classes

Without requiring the presence of order we have:

- A class \mathcal{C} of finite structures is in **NP** iff \mathcal{C} is definable in existential SOL.
- A class \mathcal{C} of finite structures is in **PH** iff \mathcal{C} is definable in SOL.

By requiring the presence of an order relation we have

- A class \mathcal{C} of finite structures is in **P** iff \mathcal{C} is 0.i. definable in existential HornESOL.
- There are similar theorems for **L**, **NL**, **PSpace**.

Numeric graph invariants (graph parameters)

We denote by $G = (V(G), E(G))$ a graph, and by \mathcal{G} and \mathcal{G}_{simple} the class of finite (simple) graphs, respectively.

A **numeric graph invariant** or **graph parameter** is a function

$$f : \mathcal{G} \rightarrow \mathbb{R}$$

which is invariant under graph isomorphism.

- (i) Cardinalities: $|V(G)|$, $|E(G)|$
- (ii) Counting configurations:
 - $k(G)$ the number of connected components,
 - $m_k(G)$ the number of k -matchings
- (iii) Size of configurations:
 - $\omega(G)$ the clique number
 - $\chi(G)$ the chromatic number
- (iv) Evaluations of graph polynomials:
 - $\chi(G, \lambda)$, the chromatic polynomial, at $\lambda = r$ for any $r \in \mathbb{R}$.
 - $T(G, X, Y)$, the Tutte polynomial, at $X = x$ and $Y = y$ with $(x, y) \in \mathbb{R}^2$.

Definability of numeric graph parameters, I

We first give examples where we use **small**, i.e., polynomial sized sums and products:

(i) The cardinality of V is FOL-definable by

$$\sum_{v \in V} 1$$

(ii) The number of connected components of a graph G , $k(G)$ is MSOL-definable by

$$\sum_{C \subseteq V: \text{component}(C)} 1$$

where $\text{component}(C)$ says that C is a connected component.

(iii) The graph polynomial $X^{k(G)}$ is MSOL-definable by

$$\prod_{c \in V: \text{first-in-comp}(c)} X$$

if we have a linear order in the vertices and $\text{first-in-comp}(c)$ says that c is a first element in a connected component.

Definability of numeric graph parameters, II

Now we give examples with possibly **large**, i.e., exponential sized sums:

(iv) The number of cliques in a graph is MSOL-definable by

$$\sum_{C \subseteq V: \text{clique}(C)} 1$$

where $\text{clique}(C)$ says that C induces a complete graph.

(v) Similarly “the number of maximal cliques” is MSOL-definable by

$$\sum_{C \subseteq V: \text{maxclique}(C)} 1$$

where $\text{maxclique}(C)$ says that C induces a maximal complete graph.

(vi) The clique number of G , $\omega(G)$ is SOL-definable by

$$\sum_{C \subseteq V: \text{largest-clique}(C)} 1$$

where $\text{largest-clique}(C)$ says that C induces a maximal complete graph of largest size.

Definability of numeric graph parameters, III

Let \mathcal{R} be a (polynomial) ring.

A numeric graph parameter $p : \text{Graphs} \rightarrow \mathcal{R}$ is **\mathcal{L} -definable** if it can be defined inductively:

- Monomials are of the form $\prod_{\bar{v}:\phi(\bar{v})} t$ where t is an element of the ring \mathcal{R} and ϕ is a formula in \mathcal{L} with first order variables \bar{v} .
- Polynomials are obtained by closing under **small products**, **small sums**, and **large sums**.

Usually, **summation** is allowed over **second order variables**, whereas **products** are over **first order variables**.

\mathcal{L} is typically **Second Order Logic** or a suitable **fragment thereof**.

We are especially interested in MSOL and CMSOL, **Monadic Second Order Logic**, possibly **augmented with modular counting quantifiers**.

If \mathcal{L} is SOL we denote the definable graphparameters by $\text{SOLEVAL}_{\mathcal{R}}$, and similarly for MSOL and CMSOL.

Our definition of **SOLEVAL** is somehow reminiscent to the definition of **Skolem's** definition of the **Lower Elementary Functions**.

References: Textbooks

- [**Pa94**] C. Papadimitriou,
Computational Complexity,
Addison-Wesley 1994
- [**EF95**] H.-D. Ebbinghaus and Jörg Flum,
Finite Model Theory,
Perspectives in Mathematical Logic,
Springer 1995, ISBN 3-540-60149-X
- [**Bo99**] B. Bollobas,
Modern Graph Theory,
Graduate Texts in Mathematics, Springer 1999.
- [**Di97**] R. Diestel,
Graph Theory,
Graduate Texts in Mathematics, Springer 1997.

Outline of the course

LECTURE 00: Second Order Logic (SOL) and its fragments (Background, not lectured)
LOGICS (14 slides)

LECTURE 01: Classical graph properties and graph parameters and their definability in
SOL (4 hours) **G-PARAMETERS**, (60 slides)

LECTURE 02: One, two, many graph polynomials (4 hours) **LANDSCAPE**, (ca. 50 slides)

LECTURE 03: The characteristic and the matching polynomial (4 hours) **MATCHING**, (54
slides)

LECTURE XX: Graph polynomials in Physics and Chemistry (2 hours) **CHEMISTRY**, (38
slides)