Intriguing Graph Polynomials,

or, why is the chromatic polynomial a polynomial?

Towards a systematic approach to graph polynomials.

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Outline

- The chromatic polynomial: G. Birkhoff 1912
- One, two, many chromatic polynomials ...
- The classical graph polynomials:
  Matching polynomial, Tutte polynomial, characteristic polynomial,
  interlace polynomial, etc, ...
- Generating functions and subset expansions
- Recursive definitions of graph polynomials and universality properties
- If time permits: Complexity (and algebraic geometry)

The Haifa Graph Polynomial Project

`http://www.cs.technion.ac.il/~janos/RESEARCH/gp-homepage.html`

The Goal:

- To discern patterns shared by many graph polynomials.
- To formulate theorems which apply to infinite classes of graph polynomials.
- To develop a general theory of graph invariants.

The Chromatic Polynomial

and

Its Variations

The (vertex) chromatic polynomial

Let $G = (V(G), E(G))$ be a graph, and $\lambda \in \mathbb{N}$.

A **λ-vertex-coloring** is a map

$$c : V(G) \to [\lambda]$$

such that $(u, v) \in E(G)$ implies that $c(u) \neq c(v)$.

We define $\chi(G, \lambda)$ to be the number of λ-vertex-colorings

Theorem: (G. Birkhoff, 1912)

$\chi(G, \lambda)$ is a polynomial in $\mathbb{Z}[\lambda]$.

Proof:

(i) $\chi(E_n) = \lambda^n$ where E_n consists of n isolated vertices.

(ii) For any edge $e = E(G)$ we have $\chi(G - e, \lambda) = \chi(G, \lambda) + \chi(G/e, \lambda)$.

Interpretation of $\chi(G,\lambda)$ for $\lambda \notin \mathbb{N}$

What's the point in considering $\lambda \notin \mathbb{N}$?

Stanley, 1973 For simple graphs G, $|\,\chi(G,-1)\,|$ counts the number of acyclic orientations of G.

Stanley, 1973 There are also combinatorial interpretations of $\chi(G,-m)$ for each $m \in \mathbb{N}$, which are more complicated to state.

Open: What about $\chi(G,\lambda)$ for each $m \in \mathbb{R} - \mathbb{Z}$?

Variations on coloring, I

We can count other coloring functions.

- proper λ-edge-colorings:

  $f_E : E(G) \to [\lambda]$ such that if $(e, f) \in E(G)$ have a common vertex then $f_E(e) \neq f_E(f)$.

  $\chi_e(G, \lambda)$ denotes the number of λ- edge-colorings

- Total colorings

  $f_V : V \to [\lambda_V]$, $f_E : E \to [\lambda_E]$ and $f = f_V \cup f_E$,
  with f_V a proper vertex coloring and f_E a proper edge coloring.

- Connected components

  $f_V : V \to [\lambda_V]$, If $(u, v) \in E$ then $f_V(u) = f_V(v)$.

- Hypergraph colorngs

  Vitaly I. Voloshin, Coloring Mixed Hypergraphs: Theory, Algorithms and Applications, Fields Institute Monographs, AMS 2002

Fact: The corresponding counting functions are polynomials in λ.

Variations on coloring, II

Let $f : V(G) \to [\lambda]$ be a function, such that Φ is one of the properties below and $\chi_\Phi(G, \lambda)$ denotes the number of such colorings with atmost λ colors.

* **convex:** Every monochromatic set induces a connected graph.

* **injective:** f is injectiv on the neighborhood of every vertex.

- **complete:** f is a proper coloring such that every pair of colors occurs along some edge.

* **harmonious:** f is a proper coloring such that every pair of colors occurs at most once along some edge.

- **equitable:** All color classes have (almost) the same size.

* **equitable, modified:** All non-empty color classes have the same size.

Fact: For (*), $\chi_\Phi(G, \lambda)$ is a polynomial in λ, for (-), it is not.

Variations on coloring, III

* **path-rainbow:** Let $f : E \to [\lambda]$ be an edge-coloring. f is **path-rainbow** if between any two vertices $u, v \in V$ there as a path where all the edges have different colors.

Fact: $\chi_{rainbow}(G, \lambda)$, the number of path-rainbow colorings of G with λ colors, is a polynomial in λ

Rainbow colorings of various kinds arise in computational biology

* **-monochromatic components:** Let $f : V \to [\lambda]$ be an vertex-coloring and $t \in \mathbb{N}$. f is an mcc_t-coloring of G with λ colors, if all the connected components of a monochromatic set have size at most t.

Fact: For fixed $t \geq 1$ the function $\chi_{mcc_t}(G, \lambda)$, the number of mcc_t-colorings of G with λ colors, is a polynomial in λ. but not in t.

mcc_t colorings were first studied in:

N. Alon, G. Ding, B. Oporowski, and D. Vertigan. Partitioning into graphs with only small components. *Journal of Combinatorial Theory, Series B*, 87:231–243, 2003.

Variations on coloring, IV

Let $\mathcal{P}$ be any graphs property and let $n \in \mathbb{N}$.

We can define coloring functions $f : V \to [\lambda]$ by requiring that the union of any n color classes induces a graph in $\mathcal{P}$.

- For $n = 1$ and $\mathcal{P}$ the empty graphs $G = (V, \emptyset)$ we get the proper colorings.
- For $n = 1$ and $\mathcal{P}$ the connected graphs we get the convex colorings.
- For $n = 1$ and $\mathcal{P}$ the graphs which are disjoint unions of graphs of size at most t, we get the mcc_t-colorings.
- For $n = 2$ and $\mathcal{P}$ the acyclic graphs we get the **acyclic colorings**, introduced in: B. Grunbaum, Acyclic colorings of planar graphs, Israel J. Math. 14 (1973), 390-412

Theorem: Let $\chi_{\mathcal{P},n}(G, \lambda)$ be the number of colorings of G with λ colors such that the union of any n color classes induces a graph in $\mathcal{P}$.

Then $\chi_{\mathcal{P},n}(G, \lambda)$ is a polynomial in λ.

Variations on colorings, V: coloring relations

Let $G = (V, E)$. Here we look at an example where the coloring is a relation $R \subseteq V \times [k]$ rather than a function $f : V \to [k]$.
We denote by C_v the set $\{c \in [k] : (v, c) \in R\}$.

Let $a, b \in \mathbb{N}$. An (a, b)-coloring relation with k colors is a relation $R \subseteq V \times [k]$ such that

- For each $v \in V$ there are at most a-many colors $c \in [k]$ such that $(v, c) \in R$.
- If $(u, v) \in E$ then $C_u \neq C_v$ and there are at most b-many distinct elements $c_1, \ldots, c_b$ in $C_u \cap C_v$.

Exercise:

- Compute the number of (a, b)-coloring relations of the complete graphs K_n for various $a, b, k \in \mathbb{N}$.
- Is the number (a, b)-coloring relations with k colors of a graph G a polynomial in a, b or k?
- Look at the corresponding definitions with "at most" replaced by "at least" or "exactly".

Variations on colorings, VI: Two kinds of colors.

Let $G = (V, E)$.
Here we look at two disjoint color sets $A = [k_1]$ and $B = [k_1 + k_2] - [k_1]$.
The colors in A are called proper colorings.
Our coloring is a function $f : V \to [k_1 + k_2] = [k]$ such that

- If $(u, v) \in E$ and $f(u) \in A$ and $f(v) \in A$ then $f(u) \neq f(v)$.

- We count the number of colorings with $k = k_1 + k_2$ colors such that k_1 colors are in A, i.e., proper.

Theorem 1 (K. Dohmen, A. Pönitz and P. Tittman, 2003)
This gives us a polynomial $P(G, k_1, k)$ in k_1 and k.

Coloring Properties

Why are there many chromatic polynomials?

Colorings

Our framework is as follows:

- Let $\mathfrak{M}$ be a finite relational τ-structure with universe M.
- Let $k \in \mathbb{N}$ and $[k] = \{0, \ldots, k-1\}$.
- Let f be an r-ary function $f : M^r \to [k]$.
- We shall look at families $\mathcal{P}$ consisting of triples of the form $(\mathfrak{M}, f, [k])$.

Coloring properties, I

A class of such triples $\mathcal{P}$ is a **coloring property** if

Extension Property: Let $n \leq k_i, i = 1, 2$ and let
$(\mathfrak{M}, f, [k_1])$ and $(\mathfrak{M}, f, [k_2])$ be two colorings of $\mathfrak{M}$,
using only colors in $[n]$, i.e., $rg(f) \subseteq [n]$.
Then $(\mathfrak{M}, f, [k_1]) \in \mathcal{P}$ iff $(\mathfrak{M}, f, [k_2]) \in \mathcal{P}$.

Isomorphism Property: $\mathcal{P}$ is closed under isomorphisms of colorings.

The isomorphisms property implies the permutation property:

Permutation Property: Let $f : M^r \to [k]$ be a fixed coloring.
For π is a permutation of $[k]$, we define the coloring f_π by $f_\pi(\bar{a}) = \pi(f(\bar{a}$.
Then $(\mathfrak{M}_k, f, [k]) \in \mathcal{P}$ iff $(\mathfrak{M}_k, f_\pi, [k]) \in \mathcal{P}$.

Coloring properties, II

If instead of coloring functions f we allow coloring relations

$$R \subseteq M^R \times [k]$$

we need some additional properties:

(i) A coloring property $\mathcal{P}$ of triples $(\mathfrak{M}_k, R, [k]) \in \mathcal{P}$ is **bounded**, if for every $\mathfrak{M}$ there is a number N_M such that for all $k \in \mathbb{N}$ the set of colors

$$\{x \in [k] : \exists \bar{y} \in M^r R(\bar{y}, x)\}$$

has size at most N_M.

(ii) A coloring property is **range bounded**, if its range is bounded in the following sense: There is a number $d \in \mathbb{N}$ such that for every $\mathcal{M}$ and $\bar{y} \in M^r$ the set $\{x \in [k] : R(\bar{y}, x)\}$ has at most d elements.

Clearly, if a coloring property is range bounded, it is also bounded.

Coloring properties, III

We denote by

$$\chi_{\mathcal{P}}(\mathfrak{M}, k)$$

the number of generalized $k-\mathcal{P}$-coloring R on $\mathfrak{M}$.

Tomer Kotek, Johann A. Makowsky, and Boris Zilber

On Counting Generalized Colorings

to appear in:

Model Theoretic Methods of Finite Combinatorics,

M. Grohe and J.A. Makowsky eds.

Contemprorary Mathematics Series of the AMS (2011)

Conference version in: Computer Science Logic, CSL'08, vol. 5213, (2008), 339-353

Uniform Definability in Logical Formalisms

Let ϕ be a sentence of some logic $\mathcal{L}$.

$\mathcal{L}$ could be first order logic **FOL**, second order logic **SOL**, monadic second order logic **MSOL**, or some fragment thereof.

We shall be interested in cases where $\mathcal{P}$ is definable in $\mathcal{L}$ by a formula $\phi(R) \in \mathcal{L}$.

If $\phi(R)$ defines a (bounded) coloring property, we say that
$\phi(R)$ is a coloring formula.

If $\mathcal{P}$ is $\mathcal{L}$-definable we call $\chi_{\mathcal{P}}(\mathfrak{M}, k)$ an $\mathcal{L}$-chromatic counting function and write

$$\chi_{\phi(R)}(\mathfrak{M}, k) = \chi_{\mathcal{P}}(\mathfrak{M}, k).$$

All the examples encountered so far are **SOL**-chromatic counting function.

Generalized multi-colorings

To construct also graph polynomials in several variables,
we extend the definition to deal with several color-sets,
and also call them generalized chromatic polynomials.

We say an $(\alpha + 2)$-tuple

$$(\mathfrak{M}, R, [k_1], \ldots, [k_\alpha],)$$

with

$$R \subset M^m \times [k_1]^{m_1} \times \ldots \times [k_\alpha]^{m_\alpha}$$

is a generalized multi-coloring of $\mathfrak{M}$ for colors $\bar{k}^\alpha = (k_1, \ldots, k_\alpha)$.

The extension and isomorphism property are adapted appropriately to deal also with unused color-sets.

By abuse of notation, $m_i = 0$ is taken to mean the color-set k_i is not used in R.

A theorem with an elementary generic proof

THEOREM:

For every $\mathcal{M}$ the counting function $\chi_{\phi(R)}(\mathcal{M}, k)$ is a polynomial in k of the form

$$\sum_{j=0}^{d \cdot |M|^m} c_{\phi(R)}(\mathcal{M}, j) \binom{k}{j}$$

where $c_{\phi(R)}(\mathcal{M}, j)$ is the number of generalised $k - \phi$-colorings R with a fixed set of j colors.

Polynomials in $\mathbb{Z}[k]$ with monomials of the form $\binom{k}{j}$
are sometimes called **Newton polynomials**.

In the light of this theorem we call $\chi_{\phi(R)}(\mathcal{M}, k)$

also a *generalised chromatic polynomial*.

Proof

We first observe that any generalised coloring R uses at most

$$N = d\cdot \mid M \mid^m$$

of the k colors.

For any $j \leq N$, let $c_{\phi(R)}(\mathcal{M}, j)$ be the number of colorings, with a fixed set of j colors, which are generalised vertex colorings and use all j of the colors.

Next we observe that any permutation of the set of colors used is also a coloring.

Therefore, given k colors, the number of vertex colorings that use exactly j of the k colors is the product of $c_{\phi(R)}(\mathcal{M}, j)$ and the binomial coefficient $\binom{k}{j}$. So

$$\chi_{\phi(R)}(\mathcal{M}, k) = \sum_{j \leq N} c_{\phi(R)}(\mathcal{M}, j) \binom{k}{j}$$

The right side here is a polynomial in k, because each of the binomial coefficients is. We also use that for $k \leq j$ we have $\binom{k}{j} = 0$. Q.E.D.

The Classical Graph Polynomials

Prominent graph polynomials

- The chromatic polynomial (G. Birkhoff, 1912)
- The Tutte polynomial and its colored versions
  (W.T. Tutte 1954, B. Bollobas and O. Riordan, 1999);
- The characteristic polynomial
  (T.H. Wei 1952, L.M. Lihtenbaum 1956, L. Collatz and U. Sinogowitz 1957)
- The various matching polynomials (O.J. Heilman and E.J. Lieb, 1972)
- Various clique and independent set polynomials (I. Gutman and F. Harary 1983)
- The Farrel polynomials (E.J. Farrell, 1979)
- The cover polynomials for digraphs (F.R.K. Chung and R.L. Graham, 1995)
- The interlace-polynomials
  (M. Las Vergnas, 1983, R. Arratia, B. Bollobás and G. Sorkin, 2000)
- The various knot polynomials (of signed graphs)
  (Alexander polynomial, Jones polynomial, HOMFLY-PT polynomial, etc)

Application of graph polynomials

There are plenty of applications of graph polynomials in

- Graph theory proper and **knot theory**;
- Chemistry and biology;
- Statistical mechanics (Potts and Ising models)
- **Social networks** and **finance mathematics**;
- Quantum physics and quantum computing

How are the **classical** graph polynomials compared?

There are various ways of comparing graph polynomials:

- By **distinctive power**:

  $P(G;\bar{X}) <_{d.p.} Q(G;\bar{X})$ if for any two graphs G_1, G_2 with $Q(G_1;\bar{X}) = Q(G_2;\bar{X})$ we also have $P(G_1;\bar{X}) = P(G_2;\bar{X})$

- By **coefficient computation**:

  $P(G;\bar{X}) <_{coeff} Q(G;\bar{X})$ if there is a function F which computes for every G the coefficients of $P(G;\bar{X})$ from the coefcients of $Q(G;\bar{X})$.

- By **substitution instance**.

  $P(G;\bar{X}) <_{subst} Q(G;\bar{X})$ if there is a substitution σ of the variables such that for every G $P(G;\bar{X}) = Q(G;\sigma(\bar{X}))$.

Proposition: $P(G;\bar{X}) <_{d.p.} Q(G;\bar{X})$ iff $P(G;\bar{X}) <_{coeff} Q(G;\bar{X})$

If $P(G;\bar{X}) <_{coeff} Q(G;\bar{X})$ then $P(G;\bar{X}) <_{subst} Q(G;\bar{X})$, but not conversely.

How are the **classical** graph polynomials **uniformly** defined?

Most graph polynomials studied in the literature have **several equivalent definitions**:

(i) By **counting generalized colorings**;

(ii) By **generating functions**;

(iii) By **subset expansion** formulas;

(iv) By **recurrence relations**;

(v) By **counting (weighted) homomorphisms**.

We shall see that, by imposing $\mathbf{SOL}$-definability,
(i)-(iii) give the same class of graph polynomials,
whereas (iv) and (v) are special cases thereof.

Complexity of the **classical** graph polynomials?

There are various problems with measuring the complexity ofa a multi-variate graph polynomial $P(G; \bar{X})$:

Turing Complexity:

Evaluation: Fix $x_0 \in \mathbb{Q}^m$. Measure the complexity of computing $P(G; x_0)$ as a function of the size of G in term is Turing complexity.

Computing the coefficients: Measure the complexity of computing the coefficients of $P(G; x_0)$ as a function of the size of G.

It is usually in EXPTIME, often $\sharp P$-complete, but sometimes in P-Time.

The Turing model does not fit the algebraic character of the problem.

BSS-Complexity: Think of a (weighted) graph being given by its adjacency matrix M_G. Measure the complexity of computing the coefficients of $P(G; \bar{X})$ from the matrix M_G.

It is usually in EXPTIME, but no convincing complexity classes fit the framework.

The matching polynomials: Case study

Two univariate matching polynomials

We denote by $m_k(G)$ the number of k-matchings of a graph G,

$$m(G,X) = \sum_{k}^{\frac{n}{2}} (-1)^k m_k(G) X^{n-2k}$$

is called the **acyclic polynomial of G** or the **matching defect polynomial**.

$$g(G,X) = \sum_{k}^{n} m_k(G) X^k$$

is called the **matching generating polynomial of G**.

- It is easy to verify the identity $m(G,X) = X^n g(G,(-X^{-2}))$
- Note that $g(G;X) <_{coeff} m(G;X)$ and hence $g(G;X) <_{d.p.} m(G;X)$, but **not conversely**.

Bivariate matching polynomial

The two matching polynomials are special cases of the **bivariate matching polynomial**

$$M(G, X, Y) = \sum_k^{\frac{n}{2}} X^{n-2k} Y^k m_k(G) = \sum_A X^{|V(G)|-2|A|} Y^{|A|}$$

where A ranges over all subsets of $E(G)$ which are matchings.

$M(G, X, Y) = \sum_k^{\frac{n}{2}} X^{n-2k} Y^k m_k(G)$ can be viewed as a **generating function**.

$M(G, X, Y) = \sum_A X^{|V(G)|-2|A|} Y^{|A|}$ can be viewed as a **subset expansion**.

Now we have $m(G; X) = M(G; X, -1)$ and $g(G; X) = M(G; 1, X)$.
In other words, both $m(G; X)$ and $g(G; X)$ are **substitution instances** of $M(G; X, Y)$.

Interpretation: $| A |$ is the size of the matching A, and $| V(G) | -2 | A |$ is the number of vertices not incident with any edge in A.

The bivariate matching polynomial as a **generalized chromatic** polynomial

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We want to show that the bivariate matching polynomial can be obtained in our framework.
We use

- two sorts of colors $[k_1]$ and $[k_2]$;
- a three-sorted structure $\mathfrak{M}_k = \langle V, [k_1], [k_2]; E, r_1, r_2 \rangle$, with two coloring relations $r_1 \subseteq E \times [k_1]$ and $r_2 \subseteq V \times [k_2]$;
- and a formula $\varphi_3(r_1, r_2)$ which says that

  (i) "$r_1 \subseteq E \times [k_1]$ is a partial function the domain M of which is a matching of G".

  (ii) and "$r_2 \subseteq V \times [k_2]$ is a partial function with domain $V - cov(M)$.

Uniform definability of subset expansions

of graph polynomials

in (Monadic) Second Order Logic SOL (MSOL)

Simple (M)SOL-graph polynomials

The graph polynomial $ind(G, X) = \sum_i ind(G, i) \cdot X^i$, can be written also as

$$ind(G, X) = \sum_{I \subseteq V(G)} \prod_{v \in I} X$$

where I ranges over all independent sets of G.
To be an independent set is definable by a formula of Monadic Second Order Logic (MSOL) $\phi(I)$.

A **simple MSOL-definable graph polynomial** $p(G, X)$ is a polynomial of the form

$$p(G, X) = \sum_{A \subseteq V(G) : \phi(A)} \prod_{v \in A} X$$

where A ranges over all subsets of $V(G)$ satisfying $\phi(A)$

and $\phi(A)$ is a (M)SOL-formula.

General (M)SOL-graph polynomials

For the general case

- One allows several indeterminates $X_1, \ldots, X_t$.
- One gives an inductive definition.
- One allows an ordering of the vertices.
- One requires the definition to be **invariant under the ordering**, i.e., different orderings still give the same polynomial.
- This also allows to define the modular counting quantifiers $C_{m,q}$ "there are, modulo q exactly m elements..."

The general case includes the Tutte and the cover polynomial, and virtually all graph polynomials from the literature.

All **SOL**- polynomials are
SOL-chromatic polynomials

Subset expansions and generalized colorings

The definition of generalized $\mathbf{SOL}$-chromatic polynomials just requires that the coloring property $\mathcal{P}$ is $\mathbf{SOL}$-definable.

THEOREM:[T. Kotek, JAM, B. Zilber]
Every $\mathbf{SOL}$-polynomial can be represented as an $\mathbf{SOL}$-chromatic polynomial, and vice versa.

Remarks:

(i) In the exact formulation of the theorem one has to be careful about the **choice of monomials**: whether they are standard, falling factorials or binomials (Newton polynomials).

(ii) The theorem fails if we replace $\mathbf{SOL}$ by $\mathbf{MSOL}$.

Recursive definition of graph polynomials

Edge eliminations

For $e = (v_1, v_2)$, we put

(i) $G - e = (V, E')$ with $E' = E - \{e\}$.
The operation of passing from G to $G - e$ is called *edge deletion*;

(ii) $G/e = (V', E')$ with $V' = V - \{v_2\}$
and $E' = (E \cap (V')^2) \cup \{(v_1, v) : (v_2, v) \in E\}$.
The operation of passing from G to G/e is called *edge contraction*.

(iii) $G \dagger e = (V', E')$ with $V' = V - \{v_1, v_2\}$ and $E' = E \cap (V')^2$.
The operation of passing from G to $G \dagger e$ is called *edge extraction*.

Recursive definitions of $\chi(G;X)$ and $M(G;X,Y)$

Let $G=(V,E)$ be a graph and $e \in E$ an edge.

$$\chi(G;X) = \chi(G-e;X) - \chi(G/e;X). \tag{1}$$

$$M(G;X,Y) = M(G-e;X,Y) + X \cdot M(G \dagger e;X,Y) \tag{2}$$

Furthermore, if $G = G_1 \sqcup G_2$ is the disjoint union of G_1 and G_2, then we have multiplicativity, i.e.

$$\chi(G_1 \sqcup G_2;X) = \chi(G_1;X) \cdot \chi(G_2;X) \tag{3}$$

$$M(G_1 \sqcup G_2;X,Y) = M(G_1;X,Y) \cdot M(G_2;X,Y) \tag{4}$$

Let $E_n = ([n],\emptyset)$. To compute the polynomials recursively we use that

$$\chi(E_n;X) = X^n \tag{5}$$

$$M(E_n,X,Y) = Y^n \tag{6}$$

Graph polynomials via recurrence relations

Other graph polynomials from the literature which satisfy similar recursive definitions are:

- the **Tutte polynomial** and its many variations and substitution instances,
- the **Cover polynomial** for directed graphs,
- and the various **Interlace polynomials**,

A **systematic study** of polynomials which are defined recursively using edge and vertex eliminations may be found in **I. Averbouch's thesis**, Haifa, 2011

Completeness and universality

Let P be a graph polynomial in a class of graph polynomials $\mathcal{S}$.

- $P \in \mathcal{S}$ is **complete for a class of graph polynomials $\mathcal{S}$** if for every other graph polynomial $Q \in \mathcal{S}$ we have $Q <_{d.p.} P$.

- $P \in \mathcal{S}$ is **universal for a class of graph polynomials $\mathcal{S}$** if for every other graph polynomial $Q \in \mathcal{S}$ we have $Q <_{subst} P$.

Invariants via recurrence relations

We can now define classes of garph polynomials via their recurrence relations

- **Chromatic invariants** (via deletion and contraction of edges)
- **Tutte-Grothendieck invariants** (via deletion and contraction of edges and case distinctions involving loops and bridges)
- EE-**invariants** with three edge eliminations
- VE-**invariants** with three vertex eliminations
- **Martin-invariants** using the operations as in the interlace polynomial

For all these classes **universal** polynomials have been identified,

by Brylawsky, Oxley; Averbouch, Godlin, JAM, Tittmann; and Courcelle.

From Recurrences to subset expansions

In the literature on the Tutte polynomials and interlace polynomial it is considered a major achievement when both a recurrence relation and a subset expansion is found.

However, under suitable, very general definability conditions, **passing from a recurrence relation to subset expansion** can always be achieved;

THEOREM: (B. Godlin, E. Katz and JAM)

Let a graph polynomial P be defined using a recurrence relation via some graph reduction operations.

Assume that the graph operations in the recurrence relation are $\mathbf{SOL}$-definable,

Then P has an **$\mathbf{SOL}$-subset expansion**.

Complexity of evaluations

The complexity of the chromatic polynomial

Theorem:

- $\chi(G,3)$ is $\sharp\mathbf{P}$-complete (Valiant 1979).
- $\chi(G,-1)$ is $\sharp\mathbf{P}$-complete (Linial 1986).
- We have an **exception set** $C = \mathbb{C} - \{0,1,2\}$ which is a countable union of semi-algebraic sets of dimension 0 in the field $\mathbb{C}$.
- We have a dichotomy: For $a \in C$ $\chi(G,a)$ is $\sharp\mathbf{P}$-hard, and for $a = \{0,1,2\}$ $\chi(G,a)$ is P-Time computable.
- Furthermore, there is a **uniform** reduction between the graph invariants $\chi(G,a_1)$ and $\chi(G,a_2)$ for $a_1, a-2$ not in C.

Question: Is there a general theorem?

The uniform difficult point property UDPP

Let $P(G, \bar{x}^m)$ be a graph polynomial in m variables.

Weak uniform DPP: $P(G, \bar{x}^m)$ has the weak uniform difficult point property (WDPP) if the following holds:

There exists an **exception set** C_Φ which is a countable union of semi-algebraic sets of dimension $< m$ in the field $\mathbb{C}$, and for all q not in the exception set C, $\Phi(-q)$ is $\sharp\mathbf{P}$ hard.

Furthermore, for any $\bar{q}_1, \bar{q}_2 \in F^m - C_P$ the graph invariants $P(G, \bar{q}_1)$ and $P(G, \bar{q}_2)$ are **uniformly algebraically reducible** to each other (say in the BSS-model of computation).

Strong uniform DPP: Additionally, all the evaluations $P(G, \bar{q})$ for $\bar{q} \in C$ are computable in polynomial time.

The Tutte polynomial

Linial and Jaeger, Vertigan and Welsh.

The paradigm of the DPP was inspired by the work on the Tutte polynomial.

(i) For the classical Tutte polynomial, the **uniform DPP** was proven by Jaeger, Vertigan and Welsh in 1990.

(ii) For the colored Tutte polynomial as defined by Bollobás and Riordan (1999), the **uniform DPP** was proven by Bläser, Dell and Makowsky in 2007.

(iii) This also holds for the multivariate Tutte polynomial, the Pott's model, if restricted to a fixed finite number of variables.

More polynomials with the uniform DPP

The uniform DPP was also proven for

(i) the cover polynomial $C(G, x, y)$ introduced by Chung and Graham (1995) by Bläser and Dell, 2007

(ii) the interlace polynomial (aka Martin polynomial) introduced by Martin (1977) and independently by Arratia, Bollobás and Sorkin (2000), by Bläser and Hoffmann, 2007

(iii) the matching polynomial and the most general EE-invariant $\xi(G, x, y, z)$, by Averbouch and Makowsky, 2007 and Hoffmann, 2010.

(iv) the harmonious chromatic polynomial, by Averbouch, Kotek and Makowsky, 2007

More cases are discussed in C. Hoffmann's PhD Thesis, Computational Complexity of Graph Polynomials, 2010, Saarbrücken.

What is the pattern behind this?

In establishing the UDPP one uses the fact that in the examples the evaluations at integer points are in $\sharp\mathbf{P}$.

We call such graph polynomials counting.

There seems to be dichtomy property:

- Either all the evaluations of a graph polynomial P are polynomial time computable, or
- P has the uniform difficult point property UDPP.

Conjecture: This dichtomy holds for all counting $\mathbf{MSOL}$-definable graph polynomials.

Note that it holds for the harmonious chromatic polynomial, which is not $\mathbf{MSOL}$-definable.

A new graph polynomial - the regular check list

Every time a new graph polynomial appears, it is natural to ask several questions about it:

(i) How can it be presented as a generalized $\mathbf{SOL}$-coloring?

(ii) How can it be presented by an $\mathbf{SOL}$-subset-expansion formula?

(iii) Does it satisfy a linear recurrence relation and is universal for it!

(iv) What is its connection to known graph polynomials?

(v) How hard is it to compute?

(vi) and finally: **Is it really new?**

Thank you for your attention !