Hankel Matrices:

From Words to Graphs

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Overview

- Hankel matrices: A brief history
- Hankel matrices in Automata Theory
- Definability in (Monadic) Second Order Logic
- Characterzing word functions
- The Finite Rank Theorem
- Meta Theorems and Hankel matrices
- Tropical semirings
- Conclusions

What are Hankel matrices?

Hankel matrices (over a field $\mathcal{F}$)

Let $f : \mathcal{F} \to \mathcal{F}$ be a function.

A finite or infinite matrix $H(f) = h_{i,j}$ over a field $\mathcal{F}$ is a Hankel matrix for f if $h_{i,j} = f(i + j)$.

Hankel matrices have many applications in:
numeric analysis, probability theory and combinatorics.

- Padé approximations
- Orthogonal polynomials
- Probability theory (theory of moments)
- Coding theory (BCH codes, Berlekamp-Massey algorithm)
- Combinatorial enumerations
  (Lattice paths, Young tableaux, matching theory)

Hankel matrices over words

Let Σ be a finite alphabet and $\mathcal{F}$ be a field and let $f : \Sigma^\star \to \mathcal{F}$ be a function on words.

A finite or infinite matrix $H(f) = h_{u,v}$ indexed over the words $u, v \in \Sigma^\star$ is a Hankel matrix for f if $h_{u,v} = f(u \circ v)$. Here $\circ$ denotes concatenation.

Hankel matrices over words have applications in

- Formal language theory and stochastic automata,
  J. Carlyle and A. Paz 1971
- Learning theory (exact learning of queries).
  A.Beimel, F. Bergadano, N. Bshouty, E. Kushilevitz, S. Varricchio 1998
  J. Oncina 2008
- Definability of picture languages.
  O. Matz 1998, and D. Giammarresi and A. Restivo 2008

Hankel matrices for graphs

If we want to define Hankel matrices for (labeled) graphs,

what plays the role of concatenation?

- Disjoint union
  Used by Freedman, Lovász and Schrijver, 2007, for characterizing multiplicative graph parameters over the real numbers

- k-unions (connections, connection matrices)
  Used by Freedman, Lovász, Schrijver and Szegedy, 2007ff, for characterizing various forms and partition functions.

- Joins, cartesian products, generalized sum-like operations
  used by Godlin, Kotek and JAM to prove non-definability.

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Hankel matrices in Automata Theory

- Probabilistic Automata
- Multiplicity Automata
- Back to overview

Probabilistic automata (Rabin 1961)

A vector $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r$ is stochastic if each $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$.

A matrix $\mu \in \mathbb{R}^{r \times r}$ is row-stochastic (column-stochasttic) if each row-vector (column-vector) is stochastic. μ is doubly stochastic if it is both row- and column-stochastic.

A Probabilistic Automaton (PA) A of size r is given by:

- A set $\{\mu_\sigma : \sigma \in \Sigma\}$ of $r \times r$ doubly stochastic matrices;
- Two stochastic vectors $\lambda, \gamma \in \mathcal{F}^r$.
- A defines a function $f_A : \Sigma^\star \to \mathbb{R}$

$$f_A(w) = f_A(\sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_n) = \lambda \mu_{\sigma_1} \mu_{\sigma_2} \cdot \ldots \cdot \mu_{\sigma_n} \gamma^t$$

- A function $f : \Sigma^\star \to \mathbb{R}$ is PA-recognizable if $f = f_A$ for some PA A.

Intuition behind probabilistic automata

- The automaton has r states.
- λ gives the probability λ_i that the automaton is in state i when reading the empty word.
- μ_σ is the transition matrix for the transition when reading σ..
- γ gives the probability γ_i that state i is an accepting state.

Multiplicity automata (Schutzenberger, 1961)

A Multiplicity Automaton (MA) A of size r over a field $\mathcal{F}$ is given by:

- A set $\{\mu_\sigma : \sigma \in \Sigma\}$ of $r \times r$ matrices over $\mathcal{F}$;
- Two vectors $\lambda, \gamma \in \mathcal{F}^r$.
- A defines a function $f_A : \Sigma^\star \to \mathcal{F}$

$$f_A(w) = f_A(\sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_n) = \lambda \mu_{\sigma_1} \mu_{\sigma_2} \cdot \ldots \cdot \mu_{\sigma_n} \gamma^t$$

- A function $f : \Sigma^\star \to \mathcal{F}$ is MA-recognizable if $f = f_A$ for some MA A.

Probabilistic automata (PA) and Multiplicity automata (MA) where introduced independently, generalizing the developments described in the famous paper by M. Rabin and D. Scott (1959).

Word functions and power series

Let $\mathcal{F}$ be a field (or semi-ring) and Σ an alphabet.

We can view Σ as a set of non-commutative indeterminates and $\Sigma^\star$ is its set of monomials.

A function $f : \Sigma^\star \to \mathcal{F}$ the defines a power series

$$S_f(w) = \sum_{w \in \Sigma^\star} f(w)w$$

A power series is rational if it can be obtained from polynomials by addition, multiplication, external products and the star-operation.

Regular languages and power series

We define a language $L(f) = \{w \in \Sigma^\star : f(w) \neq 0\}$.

$L(f)$ is FA-recognizable if there is a determinsitic finite automaton A which accepts $L(f)$.

Theorem: (Kleene-Schützenberger)

In the case of $\mathcal{F} = \mathbb{Z}_2$ the following are equivalent:

(i) $L(f)$ is FA-recognizable;

(ii) $L(f)$ is regular;

(iii) $S_f(w)$ is rational.

MA-Recognizable word functions

A function $f : \Sigma^\star \to \mathcal{F}$ is MA-recognizable if there exists an MA A such that $f_A = f$.

Theorem: (Schützenberger 1961)

For arbitrary semi-rings $\mathcal{F}$ the following are equivalent:

(i) f MA-recognizable

(ii) $S_f(w)$ is rational

Is there an analogue for **regular expressions** for MA over $\mathcal{F}$?

Multiplicity Automata and Hankel matrices (over a field)

THEOREM: (J. Carlyle and A. Paz 1971)

For a function $f : \Sigma^\star \to \mathcal{F}$ the following are equivalent:

(i) f is MA-recognizable;

(ii) S_f is rational

(iii) the Hankel matrix $H(f)$ has finite rank over $\mathcal{F}$.

This is an **ALGEBRAIC** **characterization of MA-recognizability**.

The Büchi-Elgot-Trakhtenbrot Theorem (around 1960)

A word w of size n over an alphabet Σ can be considered as a structure

$$\mathfrak{A}_w = \langle [n], <_{nat}, P_\sigma, (\sigma \in \Sigma)\rangle$$

where $P_\sigma : \sigma \in \Sigma$ is a partition of $[n]$ into possibly empty sets.

THEOREM: (R. Büchi, C. Elgot and B. Trakhtenbrot)

The following are equivalent:

(i) L is FA-recognizable;

(ii) L is regular;

(iii) The class $\{\mathfrak{A}_w : w \in L\}$ of structures is **definable in Monadic Second Order Logic**.

Is there an analogue for MA-recognizability ?

Definability of Word Functions and Graph Parameters in Monadic Second Order Logic

- The general framework of SOLEVAL
- SOLEVAL Word functions
- SOLEVAL Graph parameters and polynomials

MSOLEVAL$_{\mathcal{F}}$, I

Let $\mathcal{F}$ be a field (or a ring or a commutative semiring).
Let τ be a vocabulary (set of relation symbols and constants)/

MSOLEVAL$_{\mathcal{F}}$ consists of those functions mapping relational structures into $\mathcal{F}$ which are definable in Monadic Second Order Logic MSOL.

The functions in MSOLEVAL$_{\mathcal{F}}$ are represented as terms associating with each τ-structure $\mathcal{A}$ a polynomial $p(\mathcal{A}, \bar{X}) \in \mathcal{F}[\bar{X}]$.

Similarily, CMSOLEVAL$_{\mathcal{F}}$ is obtained by replacing MSOL by Monadic Second Order Logic with modular counting CMSOL.

MSOLEVAL$_{\mathcal{F}}$ was first studied in a sequence of papers on graph polynomials by J.A.M. co-authored with B. Courcelle, B. Godlin, T. Kotek, U. Rotics, B. Zilber.

MSOLEVAL$_{\mathcal{F}}$, II

MSOLEVAL$_{\mathcal{F}}$ is defined inductively:

(i) monomials are products of constants in $\mathcal{F}$ and indeterminates in $\bar{X}$ and the product ranges over elements a of $\mathcal{A}$ which satisfy an MSOL-formula $\phi(a)$.

(ii) polynomials are then defined as sums of monomials where the sum ranges over unary relations $U \subset A$ satisfying an MSOL-formula $\psi(U)$.

We procced now by examples of word functions in MSOLEVAL.

Examples of word functions in MSOLEVAL, I

Let $\Sigma = \{0, 1\}$ and $w \in \Sigma^*$ be represented by the structure

$$\mathcal{A}_w = \langle [\ell(w)], <, P_0, P_1 \rangle.$$

Counting occurrences:

(i) The function $\sharp_1(w)$ counts the number of occurences of 1 in a word w can be written as

$$\sharp_1(w) = \sum_{i \in [n] : P_1(i)} 1.$$

(ii) The polynomial $X^{\sharp_1(w)}$ can be written as

$$X^{\sharp_1(w)} = \prod_{i \in [n] : P_1(i)} X.$$

Examples of word functions in MSOLEVAL, II

Let L be a regular language defined by the MSOL-formula ϕ_L.

The polynomial

$$\sharp_L(w) = \sum_{u \in L : \exists v_1, v_2 (w = v_1 \circ u \circ v_2)} X^{\ell(u)}$$

is the generating function of the number of (contiguous) occurences of words $u \in L$ in a word w of size i.

It can also be written as

$$\sharp_L(w) = \sum_{U \subseteq [n] : \psi_L(U)} \prod_{i \in U} X,$$

where $\psi_L(U)$ says that U is an interval and ϕ_L^U, the relativization of ϕ_L to U holds.

Examples of word functions in MSOLEVAL, III

Let $\text{int}(w) = \sum_{i=0}^{\ell(w)-1} 2^{-i} w[i]$.

$\text{int}(w)$ considers w as a rational number in $[0, 1]$ written in binary and computes its value.

$\text{int}(w)$ can be written as

$$\text{int}(w) = \sum_{U \subset [\ell(w)] : \text{INIT}_1(U)} \prod_{i \in U} (2^{-1})$$

where $\text{INIT}_1(U)$ says that U is an initial segment of $\langle \ell(w), < \rangle$ where the last element is in P_1.

It should be clear that it is very *convenient* and *user friendly* to define word functions as terms in $\text{MSOLEVAL}_{\mathcal{F}}$.

Examples of word functions NOTin MSOLEVAL

(i) The function sqexp(w) $= 2^{\ell(w)^2} = \prod_{(x,y):x=x \wedge y=y} 2$ is not in MSOLEVAL because the product is over tuples, rather than elements.

(ii) The function dexp(w) $= 2^{2^{\ell(w)}}$ is not representable in MSOLEVAL$_{\mathcal{F}}$ due to a growth argument.

Definability of numeric graph invariants and graph polynomials

We denote by $G = (V(G), E(G))$ a graph,
and by $\mathcal{G}$ and $\mathcal{G}_{simple}$ the class of finite (simple) graphs, respectively.

A **numeric graph invariant** or **graph parameter** is a function

$$f : \mathcal{G} \to \mathbb{R}$$

which is invariant under graph isomorphism.

(i) Cardinalities: $|V(G)|$, $|E(G)|$

(ii) Counting configurations:

$k(G)$ the number of connected components,
$m_k(G)$ the number of k-matchings

(iii) Size of configurations:

$\omega(G)$ the clique number
$\chi(G)$ the chromatic number

(iv) Evaluations of graph polynomials:

$\chi(G, \lambda)$, the chromatic polynomial, at $\lambda = r$ for any $r \in \mathbb{R}$.
$T(G, X, Y)$, the Tutte polynomial, at $X = x$ and $Y = y$ with $(x, y) \in \mathbb{R}^2$.

Definability of numeric graph parameters, I

We first give examples where we use small, i.e., polynomial sized sums and products:

(i) The cardinality of V is FOL-definable by

$$\sum_{v \in V} 1$$

(ii) The number of connected components of a graph G, $k(G)$ is MSOL-definable by

$$\sum_{C \subseteq V : \mathsf{component}(C)} 1$$

where component(C) says that C is a connected component.

(iii) The graph polynomial $X^{k(G)}$ is MSOL-definable by

$$\prod_{c \in V : \mathsf{first-in-comp}(c)} X$$

if we have a linear order in the vertices and $\mathsf{first} - \mathsf{in} - \mathsf{comp}(c)$ says that c is a first element in a connected component.

Definability of numeric graph parameters, II

Now we give examples with possibly large, i.e., exponential sized sums:

(iv) The number of cliques in a graph is MSOL-definable by

$$\sum_{C \subseteq V : \text{clique}(C)} 1$$

where clique(C) says that C induces a complete graph.

(v) Similarly "the number of maximal cliques" is MSOL-definable by

$$\sum_{C \subseteq V : \text{maxclique}(C)} 1$$

where maxclique(C) says that C induces a maximal complete graph.

(vi) The clique number of G, $\omega(G)$ is is SOL-definable by

$$\sum_{C \subseteq V : \text{largest}-\text{clique}(C)} 1$$

where largest $-$ clique(C) says that C induces a maximal complete graph of largest size.

Definability of numeric graph parameters, III

Let $\mathcal{R}$ be a (polynomial) ring.

A numeric graph parameter $p : Graphs \rightarrow \mathcal{R}$ is $\mathcal{L}$-definable if it can be defined inductively:

- Monomials are of the form $\prod_{\bar{v}:\phi(\bar{v})} t$ where t is an element of the ring $\mathcal{R}$ and ϕ is a formula in $\mathcal{L}$ with first order variables $\bar{v}$.
- Polynomails are obtained by closing under small products, small sums, and large sums.

Definability of numeric graph parameters, IV

Usually, summation is allowed over second order variables, whereas products are over first order variables.

$\mathcal{L}$ is typically Second Order Logic or a suitable fragment thereof.
We are especially interested in MSOL and CMSOL, Monadic Second Order Logic, possibly augmented with modular counting quantifiers.

If $\mathcal{L}$ is SOL we denote the definable graph parameters by $\mathrm{SOLEVAL}_{\mathcal{R}}$, and similarily for MSOL and CMSOL.

Our definition of SOLEVAL is somehow reminiscent to the defintion of Skolem's definition of the Lower Elementary Functions.

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Characterizing Word Functions with MSOLEVAL

An analogue to the
Büchi-Elgot-Trakhtenbrot Theorem
for multiplicity automata.

2005/07: M. Droste and P. Gastin,
Weighted Automata and Weighted Logics,
ICALP 2005, and
Theor. Comput. Sci., 380.1-2(2007), pp. 69-86

2013: N. Labai and J.A. Makowsky,
Weighted Automata and Monadic Second Order Logic,
Fourth International Symposium on Games, Automata, Logics and Formal Verification, GandALF 2013

Characterizing functions defined by Multiplicity Automata

Theorem: (M. Droste and P. Gastin 2005, N. Labai and J.A.M., 2012)
Let $\mathcal{F}$ be a field, and $f : \Sigma^* \to \mathcal{F}$.
The following are equivalent:

(i) $f = f_A$ for some Multiplicity Automaton A over $\mathcal{F}$.

(ii) $f \in \mathrm{MSOLEVAL}_{\mathcal{F}}$

(iii) $f \in \mathrm{CMSOLEVAL}_{\mathcal{F}}$

(iv) $M(\circ, f)$ has finite rank.

Proof: (i) $\leftrightarrow$ (iv) is the Carlyle-Paz Theorem.
(ii) $\leftrightarrow$ (iii) follows from CMSOL equals MSOL on words.
(iii) $\rightarrow$ (iv) is the Finite Rank Theorem.
(i) $\rightarrow$ (ii) is proven using matrix algebra and logic.

Skip comparison

Weighted RMSOL vs. MSOLEVAL, I

There are **notational disadvantages** in the Droste/Gastin approach with RMSOL.

(i) The definition of RMSOL is not a purely syntactic.

(ii) The formulas are hybrid objects, mixing constants from $\mathcal{F}$ and logical expressions. For instance $\forall x \cdot 2$ is a weighted formula (for $2 = 1 + 1$ in a field) which represents the function $2^{\ell(w)}$, and $\forall x \forall y \cdot 2$ is a weighted formula which represents the function $2^{2^{\ell(w)}}$.

(iii) Seemingly equivalent formulas can represent different functions: $\exists x P_1(x)$ represents the function $\sharp_1(w)$ but $\exists(P(x) \vee P(x))$ represents the function $2 \cdot \sharp_1(w)$.

(iv) Some of these disadvantages have been corrected in very recent papers.
M. Droste and P. Gastin in the Handbook and
B. Bollig, P. Gastin , B. Monmege and M. Zeitoun presented at ICALP 2010.

Weighted RMSOL vs. MSOLEVAL, II

In contrast to these disadvantages,
$\mathrm{MSOLEVAL}_{\mathcal{F}}$ has the following advantages:

(i) The expressions are natural and intuitive.

(ii) The expressions are defined for all formulas of MSOL without any restrictions.

(iii) If we replace formulas occurring in an expression by equivalent formulas, the word function it represents remains the same.

(iv) $\mathrm{MSOLEVAL}_{\mathcal{F}}$ was used since 2000 in the study of Metatheorems and Graph polynomials.

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Hankel Matrices

and the

Finite Rank Theorem

General Hankel Matrices: I

Let $\mathcal{C}$ be a class possibly labeled graphs, hyper-graphs or τ-structures.

Let $\Box$ be a binary operation define on $\mathcal{C}$.

Let G_i be an enumeration of all (labeled) finite graphs

Let f be graph parameter.

The (full) Hankel matrix $M(f, \Box)$ is defined by

$$M(f, \Box)_{i,j} = f(G_i \Box G_j)$$

and is called the **Full Hankel Matrix of f for $\Box$ on $\mathcal{C}$**,

or just a **Hankel matrix**.

We shall often look at infinite submatrices of $M(f, \Box)$.

Logics

In this talk a logic $\mathcal{L}$ is a fragment of Second Order Logic SOL.

Let $\mathcal{L}$ be a subset of SOL. $\mathcal{L}$ is a fragment of SOL if the following conditions hold.

(i) For every finite relational vocabulary τ the set of $\mathcal{L}(\tau)$ formulas contains all the atomic τ-formulas and is closed under boolean operations and renaming of relation and constant symbols.

(ii) $\mathcal{L}$ is equipped with a notion of quantifier rank and we denote by $\mathcal{L}_q(\tau)$ the set of formulas of quantifier rank at most q. The quantifier rank is subadditive under substitution of subformulas,

(iii) The set of formulas of $\mathcal{L}_q(\tau)$ with a fixed set of free variables is, up to logical equivalence, finite.

(iv) Furthermore, if $\phi(x)$ is a formula of $\mathcal{L}_q(\tau)$ with x a free variable of $\mathcal{L}$, then there is a formula ψ logically equivalent to $\exists x \phi(x)$ in $\mathcal{L}_{q'}(\tau)$ with $q' \geq q + 1$.

(v) A fragment of SOL is called **tame** if it is closed under scalar transductions.

Typical fragments

- First Order Logic FOL.
- Monadic Second Order Logic MSOL.
- Logics augmented by modular counting quantifiers: $D_{m,i}x\phi(x)$ which says that the numbers of elements satisfying ϕ equals i modulo m.
- CFOL, CMSOL denote the logics FOL, resp. MSOL, augmented by all the modular counting quantifiers.
- Logics augmented by Lindström quantifiers.
- Logics restricted a fixed finite set of bound or free variables.

$\mathcal{L}$-smooth operations.

Let $\mathcal{L}$ be a logic.

We say that two graphs G, H are $(\mathcal{L},)q$-equivalent, and write $G \sim_{\mathcal{L}}^{q} H$, if G and H satisfy the same $\mathcal{L}$-sentences of quantifier rank q.

We say that $\Box$ is $\mathcal{L}$-smooth, if wwhenever we have

$$G_i \sim_{\mathcal{L}}^{q} H_i, i = 0, 1$$

then

$$G_0 \Box G_1 \sim_{\mathcal{L}}^{q} H_0 \Box H_1$$

This definition can be adapted to k-ary operations for $k \geq 1$.

Proving that an operation $\Box$ is $\mathcal{L}$-smooth may be difficult.

For FOL this can be achieved using

Ehrenfeucht-Fraïssé games also know as pebble games.

Anther way of establishing smoothness is via the Feferman-Vaught theorem.

Examples of $\mathcal{L}$-smooth operations, I

(i) Quantifier-free **scalar transductions** are both FOL and MSOL-smooth.

(ii) Quantifier-free **vectorized transductions** are FOL but not MSOL-smooth.

(iii) The (rich) disjoint union is both FOL and MSOL-smooth.

The **rich** disjoint union has two additional unary predicates to distinguish the universes.

For FOL this was shown by E. Beth in 1952.
For MSOL this is due to H. Läuchli, 1966, using **Ehrenfeucht-Fraïssé games**

(iv) Sum-like operations are obtained from rich disjoint unions using quantifier-free **scalar** transductions.
Sum-like operations are MSOL-smooth.

Examples of $\mathcal{L}$-smooth operations, II

(i) The cartesian product is FOL-smooth but not MSOL-smooth.
It can be obtained from the rich disjoint union
unions using quantifier-free **vectoriced** transductions.

This was shown by A. Mostowski in 1952.

(ii) Product-like operations are obtained from rich disjoint unions using quantifier-free **vectoriced** transductions.
Product-like operations are FOL-smooth but not MSOL-smooth.

(iii) Adding modular counting quantifiers to a logic $\mathcal{L}$ preserves $\mathcal{L}$-smoothness.

For CMSOL and the **disjoint union** this is due to B. Courcelle, 1990.
For $CFOL$ and the **product** this is due to T. Kotek and J.A.M., 2012.

The Finite Rank Theorem

THEOREM (Godlin, Kotek, Makowsky 2008):

Let f be a numeric parameter or polynomial for τ-structures definable in $\mathcal{L}$ and taking values in an integral domain $\mathcal{R}$.

Let $\Box$ be an $\mathcal{L}$-smooth operation.

Then the Hankel matrix $M(f, \Box)$ has finite rank over $\mathcal{R}$.

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The **Proof** uses a Feferman-Vaught-type theorem for graph polynomials, due to B. Courcelle, J.A.M. and U. Rotics, 2000.

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Finite Hankel rank vs. definability

Proposition:(after an idea of E. Specker, 1982)

- There are **only countably many** graph parameters $\mathfrak{p} \in$ CMSOLEVAL.
- Let $A \subseteq \mathbb{N}$. Let Clique$_A$ be the graph property which says that G is a clique of size $k \in A$.
- There are **continuum many** distinct graph properties of the form Clique$_A$.
- The Hankel rank of $H(\text{Clique}_A, \sqcup)$ is 1, because the disjoint union of two graphs cannot be a clique.

Metatheorems

Metatheorems relate definability criteria of concepts
to their mathematical properties (or vice versa):

Folklore: Every real function f definable by a polynomial expression is continuous.

Birkhoff's HSP Theorem: A class of algebras $\mathcal{A}$ is definable by a set of equational identities iff $\mathcal{A}$ is closed under homomorphisms, subalgebras and direckt products.

Mal'cev's Theorem: A class of algebras $\mathcal{A}$ is definable by a set of conditional equations iff $\mathcal{A}$ is closed under subalgebras and direckt products.

Büchi, Elgot, Trakhtenbrot: A language $\mathcal{L}$ is recognizable by a finite automaton iff $\mathcal{L}$ is definable in Monadic Second Order Logic

Robertson, Seymour, Courcelle: Every minor closed class of finite graphs is definable in Monadic Second Order Logic.

Notions of Width for Graphs

The width of a graph is a numeric graph parameter which measures (somehow) how far a given graph G is from highly structured graphs in a class $\mathcal{W}$.
Lower width means closer to $\mathcal{W}$.
We do not need the exact definitions for this talk.

Tree-width: $\mathcal{W}$ is the disjoint unions of trees.
A graph G has tree-with $\text{tw}(G) = 1$ iff G is a forest.
A class of finite graphs $\mathcal{K}$ is of bounded tree-width (BTW) if there is $k \in \mathbb{N}$ such that for every $G \in \mathcal{K}$ we have $tw(G) \leq k$.

Clique-width: $\mathcal{W}$ is the disjoint unions of cliques.
If a graph G is a clique, then it has clique-with $\text{cw}(G) = 2$.
A class of finite graphs $\mathcal{K}$ is of bounded clique-width (BCW) if there is $k \in \mathbb{N}$ such that for every $G \in \mathcal{K}$ we have $cw(G) \leq k$.

If $\mathcal{K}$ is BTW then it is also BCW.

Let $k_0 \in \mathbb{N}$ be fixed.
For every $k \in \mathbb{N}$ there are graphs G with $tw(G) = k$ and $cw(G) = k_0$.

Connection matrices for tree-width and clique-width

We look at the following binary operations on labeled graphs.

$G \sqcup_k H$:

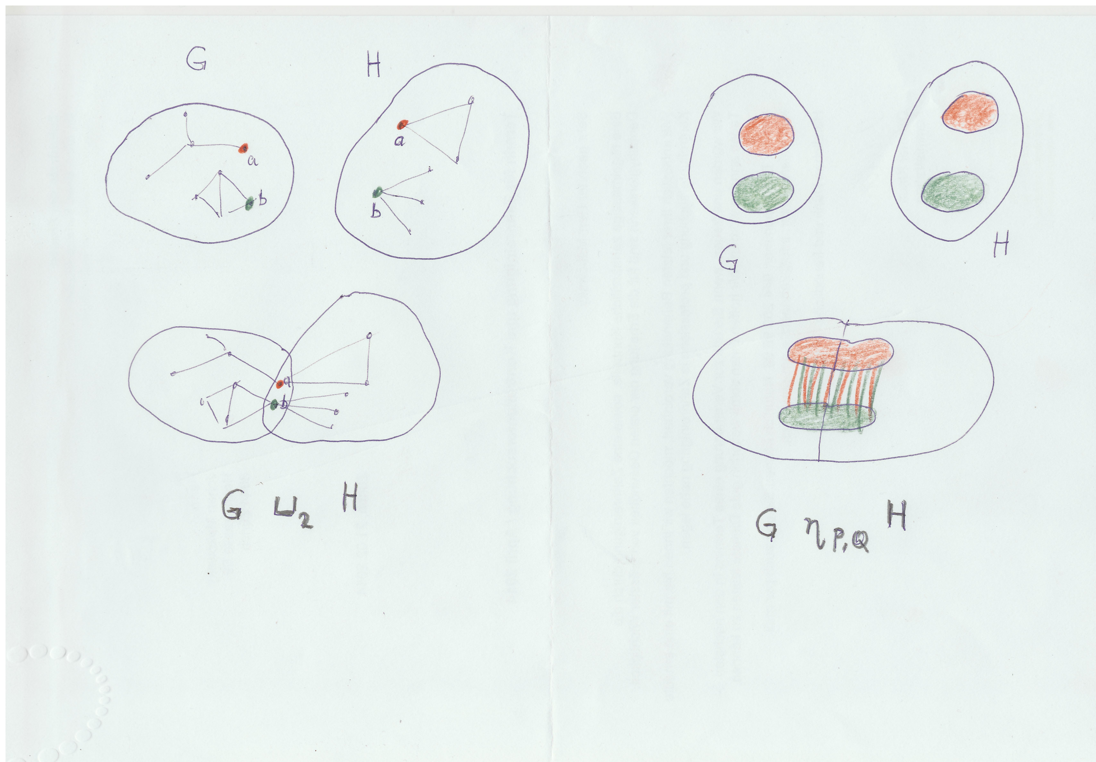
Input: Two graphs G, H with k vertices distinctly labeled.
Operation: Disjoint union with vertices of corresponding labels indetified.

$G \eta_{P,Q} H$:

Input: Two graphs G, H with two subsets each, P_G, Q_G and P_G, Q_H.
Operation: Disjoint union with additional edges connection all vertices from $P_{G \sqcup H}$ with $Q_{G \sqcup H}$.

Both operations are CMSOL-smooth and sum-like.

Identifying labeled vertices a and b Adding all the edges between P and Q



Metatheorems for tree-width: The Theorems of Courcelle and Lovász

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Let $\mathcal{K}$ be of bounded tree-width (BTW), $G \in \mathcal{K}$, and
$\mathfrak{p}$ be a graph parameter with values in a field.

Courcelle 1990:

If $\mathfrak{p}$ is boolean-valued and definable in CMSOL,
$\mathfrak{p}(G)$ is computable in polynomial (even linear) time

Courcelle, JAM, Rotics 2000:

If $\mathfrak{p}$ is real-valued and in CMSOLEVAL,

$\mathfrak{p}(G)$ is computable in polynomial (even linear) time.

Lovász 2006:

If $\mathfrak{p}$ is real-valued,
and the **Hankel matrix** $H(\mathfrak{p}, \sqcup_k)$ has finite rank,
then $\mathfrak{p}(G)$ is computable in polynomial time.

Metatheorems for clique-width

Let $\mathcal{K}$ be of bounded clique-width (BCW), and
$\mathfrak{p}$ be a graph parameter with values in a field.

Courcelle, JAM, Rotics 2000:

If $\mathfrak{p}$ is boolean and definable in CMSOL,

or real-valued and $\mathfrak{p} \in$ CMSOLEVAL,

$\mathfrak{p}(G)$ is computable in polynomial (even linear) time.

JAM, Labai 2014:

If $\mathfrak{p}$ is real-valued,

and the **Hankel matrix** $H(\mathfrak{p}, \eta_{P,Q})$ has finite rank,

$\mathfrak{p}(G)$ is computable in polynomial time.

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Skip inductive classes

Sum-like Inductive Classes of Structures

Skip inductive classes

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Sum-like inductive classes of structures, I

CMSOL-inductive classes of graphs are a generalization of graph classes of given tree-width or clique-width, and were introduced by JAM in 2004.

Special cases of CMSOL-inductive classes are the *sum-like inductive classes*.

A class of τ-structures $\mathcal{C}$ is *sum-like inductive* if, given

- a finite set of basic labeled graphs $G_j, j \leq J$,
- and a finite set of sum-like binary operations $\square_i, i \leq I$.

We define $\mathcal{C}$ inductively by

- each $G_j, j \leq J$ is in $\mathcal{C}$,
- and whenever $H_1, H_2 \in \mathcal{C}$ then also $\square_i(H_1, H_2) \in \mathcal{C}$ for all $i \leq I$.
- $\mathcal{C}$ is $\mathcal{L}$-smooth inductive, if all the operations involved are $\mathcal{L}$-smooth.

Sum-like inductive classes of structures, II

Theorem:(JAM, 2004)

(i) The classes of graphs of fixed tree-width k is sum-like inductive, using $\sqcup$, $\sqcup_i : i \leq k$ and relabeling operations $\rho_{i,j} : i < j \leq k$ for labeled vertices.

(ii) The classes of graphs of fixed clique-width k is sum-like inductive, using $\sqcup$, $\eta_{P_i \to P_j} : i, j \leq k$ and recoloring operations $\bar{\rho}_{i,j} : i < j \leq k$ for colored sets vertices.

Other examples of sum-like inductive classes of labeled graphs can be found using various graph grammars, as studied in A. Glikson's MSc Thesis (2003).

We do not know whether every $\mathcal{L}$-smooth operation on τ-structures is also sum-like.

Sum-like inductive classes of structures, III

Theorem:(Courcelle and JAM, 2002; Adler and Adler, 2008)

(i) A graph class $\mathcal{C}$ is of bounded clique-with iff it is sum-like inductive.

(ii) For general τ-structures this is not the case.

Theorem:(JAM, 2004/14)

Let $\mathcal{C}$ be a sum-like inductive class of τ-structures,
and $\mathfrak{p} \in$ CMSOLEVAL be a graph parameter.
Then $\mathfrak{P}$ can be computed on graphs $G \in \mathcal{C}$ in polynomial time in the size $pt(G)$ of the parse tree which shows that $G \in \mathcal{C}$.

In 2004 I stated the theorem for MSOL-smooth operations. However, the proof I had in mind then only works for sum-like operations.

Linearily linked Hankel matrices

Let $\Box_i, i \leq I$ be finitely many binary operations on labeled graphs, and let $G_j, j \leq J$ be a finite set of basic graphs. $\mathfrak{p}_k, k \leq K$ be finitely many real-valued graph parameters. For a labeled graph H let $\bar{\mathfrak{p}}(H)$ denote the vector $(\mathfrak{p}_1(H), \ldots, \mathfrak{p}_K(H))$.

(i) Assume $\mathcal{C}$ is *inductively defined using* $G_j, j \in J$ *and* $\Box_i, i \leq I$:

Each $G_j, j \in J$ is in $\mathcal{C}$, and whenever $H_1, H_2 \in \mathcal{C}$ then also $\Box_i(H_1, H_2) \in \mathcal{C}$.

Here $\Box_i$ does not have to be sum-like.

(ii) The Hankel matrices $H(\mathfrak{p}_k, \Box_i, i \leq I, j \leq J$ are **linearly linked** if the following hold:

(ii.a) For each $\mathfrak{p}_k, k \leq K$ and $\Box_i, i \leq I$ the Hankel matrices $H(\mathfrak{p}_k, \Box_i)$ are of finite rank.

(ii.b) For each $i \leq I$ there is a matrix P_i such that for all graphs H_1, H_2

$$\bar{\mathfrak{p}}(\Box_i(H_1, H_2)) = P_i \cdot \bar{\mathfrak{p}}(\Box_1(H_1, H_2))$$

Main Theorem

Main Theorem:(Labai and JAM, 2014)

Let $\mathcal{C}$ be *inductively defined using* $G_j, j \in J$ *and* $\Box_i, i \leq I$.

Let $\mathfrak{p}_k, k \leq K$ be finitely many graph parameters, such that the Hankel matrices $H(\mathfrak{p}_k, \Box_i), i \leq I, j \leq J$ are linearly linked.

Then for graphs $H \in \mathcal{C}$ with parse-tree $pt(H)$, all the graph parameters $\mathfrak{p}_k, k \leq K$ can be computed in **polynomial time** in the size of of the parse tree $pt(H)$ which shows that $H \in \mathcal{C}$.

Tropical commutative semirings

Let $\mathcal{T}_{min} = \langle \mathbb{R} \cup \{\infty\}, \min, +, 0, 1, \infty \rangle$ the structure consisting of the real $\mathbb{R}$ augmented with a new element ∞, min the usual minimum and $+$ the usual addition, extended with the obvious rules for ∞:

- Every real is smaller then ∞.
- For every real $a \in \mathbb{R}$, $a + \infty = \infty$

$\mathcal{T}_{min}$ is a commutative semiring with $+$ as multiplication and min as addition.

$\mathcal{T}_{min}$ is called the tropical semiring or the $(\min, +)$-algebra.

Matrices are defined in the ususal way, but rank of a matrix is more complicated, as several options to define it are not necessarily equivalent.

In the case of weighted automata with weights in an arbitrary semiring the characterization theorem works as well.

In the case graph parameters with weights in a tropical semiring the theorems do generalize as well.

However, in the case of arbitrary semirings, we only proved the meta-theorem for bounded linear clique-width. tropical

Conclusions

- We have shown that the formalism of CMSOLEVAL captures the notion of definability for graph parameters and leads to various **meta-theorems**.
- We have shown that graph parameters definable in CMSOLEVAL have finite rank Hankel matrices for **sum-like graph operations**.
- We have shown how to use finite rank Hankel matrices in order to prove meta-theorems **without definability assumptions**.

What remains be done?

Develop the **theory** and **applications** of finite rank Hankel matrices further.

Back to overview

Thank you for your attention!